Amme 3500: System Dynamics and Control

Transient Performance and the S-Plane

Dr. Ian R. Manchester
# Course Outline

<table>
<thead>
<tr>
<th>Week</th>
<th>Date</th>
<th>Content</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6 Mar</td>
<td>Introduction</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>13 Mar</td>
<td>Frequency Domain Modelling</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20 Mar</td>
<td>Transient Performance and the s-plane</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>27 Mar</td>
<td>Block Diagrams</td>
<td>Assign 1 Due</td>
</tr>
<tr>
<td>5</td>
<td>3 Apr</td>
<td>Feedback System Characteristics</td>
<td>No Friday Tutorial</td>
</tr>
<tr>
<td>6</td>
<td>17 Apr</td>
<td>Root Locus</td>
<td>Assign 2 Due</td>
</tr>
<tr>
<td>7</td>
<td>24 Apr</td>
<td>Root Locus 2</td>
<td>No Wed/Thu Tutorial</td>
</tr>
<tr>
<td>8</td>
<td>26 Apr</td>
<td>Bode Plots</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>3 May</td>
<td>Bode Plots 2</td>
<td>Assign 3 Due</td>
</tr>
<tr>
<td>10</td>
<td>10 May</td>
<td>State Space Modeling</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>17 May</td>
<td>State Space Design Techniques</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>24 May</td>
<td>Advanced Control Topics</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>31 May</td>
<td>Review</td>
<td>Assign 4 Due</td>
</tr>
<tr>
<td>14</td>
<td></td>
<td>Spare</td>
<td></td>
</tr>
</tbody>
</table>
Today’s Goals

• We continue to investigate the relationship between system components and time response, focusing on the step response

• Predict the time response of a system from its transfer function
  – The location of the poles and zeros of the transfer function in the “s-plane”
  – Special facts about 1^{st} and 2^{nd} order systems
Performance Specs

- Autonomous systems usually consist of many control loops, with each part affecting performance of the whole.
Performance Specs

- E.g. for Big Dog, stability depends on the legs reaching a certain position rapidly (within say a 10\textsuperscript{th} of a second) by choosing a torque for the system:

\[ J\ddot{\theta} + D\dot{\theta} + K\theta = T \]

- The torque is set by an inner control loop, which moves a hydraulic valve to achieve the right pressure in a cylinder within a few milliseconds.

- On an even faster timescale, a controller sets a current in a solenoid (electromagnet) which moves the valve.
Step Response

- Important characteristics of time response: final value, rise time, overshoot, settling time, oscillation...
Response Vs. Pole Location

• As we saw previously, a qualitative understanding of the effect of poles and zeros on the system response can help us to quickly estimate performance.
Response Vs. Pole Location

• We would now like to gain a deeper insight into the system performance as a function of pole and zero locations
• This will help us to describe the system performance quantitatively
• We will also see how this allows us to design a system to meet certain design constraints
General First Order System

- A first order system without zeros can be described by:
  
  \[ H(s) = \frac{a}{s + \sigma} \]

- The resulting impulse response is
  
  \[ h(t) = ae^{-\sigma t} \]

- When
  
  - \( \sigma > 0 \), pole is located at \( s < 0 \), exponential decays = stable
  
  - \( \sigma < 0 \), pole is at \( s > 0 \), exponential grows = unstable

Step Response

\[ y(t) = \frac{a}{\sigma} \left(1 - e^{-\sigma t}\right) \]
Example First Order System

• Consider heat flow from outside to inside through an insulator

\[ \dot{T}_{in} = C(T_{out} - T_{in}) \]

• Laplace transform with zero initial conditions:

\[ sT_{in} = C(T_{out} - T_{in}) \]

\[ \frac{T_{in}}{T_{out}} = \frac{C}{s + C} \]

• Pole at \( s = -C \), stable, step response:

\[ T(t) = 1 - e^{-Ct} \]
We define the *time constant* of the system as
\[ \tau = \frac{1}{\sigma} \]
This is the time taken for the system to decay to \(1/e\) or 37% of its initial value or rise to 63% of step response.
General First Order System

- The *Rise Time* $T_r$ is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

\[
y(t) = 1 - e^{-\sigma t} \\
0.9 = 1 - e^{-\sigma t_{0.9}} \\
0.1 = 1 - e^{-\sigma t_{0.1}} \\
T_R = t_{0.9} - t_{0.1} \\
= \frac{2.31}{\sigma} - \frac{0.11}{\sigma} \\
= \frac{2.2}{\sigma}
\]
The *Settling Time* $T_s$ is defined as the time for the waveform to reach and stay within a certain percentage of its final value.

Values of 1%, 2% and 5% are often used.

\[ 0.98 = 1 - e^{-\sigma T_s} \]

\[ T_s = \frac{4}{\sigma} \] (2%)

\[ T_s = \frac{4.6}{\sigma} \] (1%)

\[ T_s = \frac{3}{\sigma} \] (5%)
Example

- A system has a transfer function
  \[ G(s) = \frac{50}{s + 50} \]
- The time constant is:
  - \( \frac{1}{\sigma} = 0.02 \)
- Settling time (2%) is:
  - \( \frac{4}{\sigma} = 0.08 \)
- Rise Time is:
  - \( \frac{2.2}{\sigma} = 0.044 \)
General Second Order System

- Many systems of interest are of higher order
- For a first order system we have a single, real pole
- Second order systems are quite common and are generally written in the following standard form

\[ G(s) = C \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]
Natural Frequency and Damping Ratio

- The *Natural Frequency*, $\omega_n$, of a second-order system is the frequency of oscillation of the system without damping.

- The *Damping Ratio*, $\zeta$, of a second-order is the ratio of the exponential decay frequency and the natural frequency.
General Second Order System

- For a second order system, we can have four distinct combinations of real and imaginary poles:
  - Two real poles \(-\sigma_1, -\sigma_2\)
  - Two identical real \(-\sigma_1\)
  - Two complex poles \(-\sigma \pm j\omega_d\)
  - Two imaginary \(\pm j\omega_d\)
Natural Frequency and Damping Ratio

• How does this relate to the pole location in terms of their real and imaginary parts?

\[ s = -\sigma \pm j\omega_d \]

• Complex poles are always in complex conjugate pairs so

\[
d(s) = (s + \sigma - j\omega_d)(s + \sigma + j\omega_d)
\]

\[ = (s + \sigma)^2 + \omega_d^2 \]
Natural Frequency and Damping Ratio

- By multiplying out and comparing the denominator of the two forms we find that

\[(s + \sigma)^2 + \omega_d^2 = s^2 + 2\zeta \omega_n s + \omega_n^2\]

\[\sigma = \zeta \omega_n\]

\[\omega_d = \omega_n \sqrt{1 - \zeta^2}\]
Natural Frequency and Damping Ratio

- We can now relate the natural frequency and damping ratio to the $s$-plane:

\[
\sigma = \zeta \omega_n
\]

\[
\omega_d = \omega_n \sqrt{1 - \zeta^2}
\]

\[
\theta = \sin^{-1} (\zeta)
\]
Natural Response of Second Order System

• We’d like to find the natural response of a general 2nd order system

\[ \ddot{y} + 2\zeta \omega_n \dot{y} + \omega_n^2 y = 0 \]

\[ s^2 Y(s) - sy_0 - \dot{y}_0 + 2\zeta \omega_n (sY(s) - y_0) + \omega_n^2 Y(s) = 0 \]

\[ Y(s) = \frac{(s + 2\zeta \omega_n) y_0 + \dot{y}_0}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

• For the case of zero initial velocity we have

\[ C(s) = y_0 \frac{s + 2\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]
Natural Response of Second Order System

- We’d like to find the natural response of a general 2nd order system
- After completing squares

\[ C(s) = y_0 \frac{s + 2\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

\[ = y_0 \frac{s + \xi \omega_n + \frac{\zeta}{\sqrt{1 - \xi^2}} \omega_n \sqrt{1 - \xi^2}}{(s + \xi \omega_n)^2 + \omega_n^2 (1 - \xi^2)} \]

\[ C(s) = y_0 \frac{(s + \sigma) + \frac{\zeta}{\omega_d} \omega_d}{(s + \sigma)^2 + \omega_d^2} \]

\[ y(t) = y_0 e^{-\sigma t} \left( \cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) \]
Natural Response of Second Order System

\[ y(t) = y_0 e^{-\sigma t} \left( \cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) \]

\[ y \left( \frac{2\pi n}{\omega_d} \right) = y_0 e^{-\xi \omega_d \frac{2\pi n}{\omega_d}} \]

\[ y_n = y_0 e^{-\xi \omega_d \frac{2\pi n}{\omega_d}} \]

\[ \ln \left( \frac{y_n}{y_0} \right) = -\xi \omega_n \frac{2\pi n}{\omega_d} \]

\[ \frac{\xi \omega_n}{\omega_n \sqrt{1 - \xi^2}} = \frac{1}{2\pi n} \ln \left( \frac{y_0}{y_n} \right) \]

\[ \xi = \frac{1}{2\pi n} \ln \left( \frac{y_0}{y_n} \right) \]

• We can use this to estimate the system parameters
• By displacing the system and measuring its response we can identify the peaks and estimate natural frequency and damping ratio
• This is known as the log decrement
A Familiar Mechanical Example

- Earlier, we considered this mechanical system
- Now we can consider the natural and forced response

\[ M\ddot{y}(t) + K_d \dot{y}(t) + Ky(t) = f(t) \]

\[ M(s^2Y(s) - sy(0) - \dot{y}(0)) + K_d (sY(s) - y(0)) + KY(s) = F(s) \]
Example: Natural Response

• The natural response of the system can be found by assuming no input force

\[
(M \ddot{s} + K_d s + K)Y(s) = (M s + K_d)\nu(0) + My(0)
\]

\[
Y(s) = \frac{(M s + K_d)\nu(0) + My(0)}{(M \ddot{s} + K_d s + K)}
\]

• Taking inverse Laplace transform will give us the system response
Example: Natural Response

• Suppose we wish to find the natural response of the system to an initial displacement of 0.5m

• Assume the mass of the system is 1kg and the systems constant $k_d$ is 2 Nsec/m and $k$ is 20 N/m
Example: Natural Response

- The natural response of the system can be found by assuming no input force

\[ Y(s) = y(0) \frac{s + \frac{K_d}{M}}{s^2 + K_d \frac{s}{M} + \frac{K}{M}} = y_0 \frac{s + 2\xi \omega_n}{s^2 + 2\xi \omega_n s + \omega_n^2} \]

\[ 2\xi \omega_n = \frac{K_d}{M}, \quad \omega_n^2 = \frac{K}{M} \]
Example: Natural Response

- Substituting values, we find

\[ 2\zeta \omega_n = 2, \omega_n^2 = 20, \]
\[ \omega_n = \sqrt{20}, \zeta = \frac{1}{\sqrt{20}}, \omega_d = \sqrt{19} \]

\[ y(t) = y_0 e^{-\sigma t} \left( \cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) \]
\[ = 0.5 e^{-t} \left( \cos \sqrt{19} t + \frac{1}{\sqrt{19}} \sin \sqrt{19} t \right) \]
Example: Natural Response

- We can work backwards from the response to estimate the parameters

\[
\omega_n \approx \omega_d = \frac{2 \text{ cycles}}{2.88 s} = 0.69 \text{Hz} = 4.36 \frac{\text{rad}}{s}
\]

\[
\zeta \approx \frac{1}{2\pi n} \ln \left( \frac{y_0}{y_n} \right) = \frac{1}{4\pi} \ln \left( \frac{0.5}{0.0279} \right) = 0.230
\]
We’d like to find the step response of the standard form

After some partial fraction expansion and completing squares

Take Inverse Laplace and simplify

\[ C(s) = \frac{\omega_n^2}{s \left( s^2 + 2\zeta \omega_n s + \omega_n^2 \right)} \]

\[ = \frac{1}{s} - \frac{\zeta \omega_n}{\sqrt{1 - \zeta^2}} \left( s + \zeta \omega_n \right) + \frac{\xi}{\omega_n \sqrt{1 - \xi^2}} \]

\[ c(t) = 1 - e^{-\sigma t} \left( \cos \omega_d t + \frac{\sigma}{\omega_d} \sin \omega_d t \right) \]

\[ \sigma = \frac{1}{\sqrt{1 - \zeta^2}} \]

\[ \omega_d = \sqrt{\omega_n^2 - \sigma^2} \]
Step Response vs. Damping Ratio

- Damping ratio determines the characteristics of the system response.
Step Response vs. Damping Ratio

\[ c(t) = A \cos(\omega_n t - \phi) \]

\[ c(t) = Ae^{-\sigma_d t} \cos(\omega_d t - \phi) \quad 0 < \zeta < 1 \]

\[ c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t} \quad \zeta = 1 \]

\[ c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t} \quad \zeta > 1 \]
The damping ratio describes the degree of damping in the system.

For an underdamped system, rising damping will lower the overshoot of the system.
Time Domain Specifications

- Specifications for a control system design often involve requirements associated with the time response of the system
  - *Peak Time*, $t_p$, is the time taken to reach maximum overshoot
  - *Overshoot*, $M_p$, is the maximum amount the system overshoots its final value
  - *Settling time*, $t_s$, is the time for system transients to decay
  - *Rise time*, $t_r$, is the time it takes for the system to reach the vicinity of its new set point
Time Domain Specifications

- Rise time, settling time and peak time yield information about the speed of response of the transient response.
- This can help a designer determine if the speed and nature of the response is appropriate.
Peak Time

- Peak time is found by differentiating and finding the first zero crossing after $t=0$.

\[ L[\dot{c}(t)] = sC(s) \]
\[ = \frac{\omega_n^2}{s^2 + 2\zeta \omega_n + \omega_n^2} \]

- After some manipulation we find $t_p$ occurs when $\sin\omega_d t = 0$.

\[ t_p = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}} \]
\[ = \frac{\pi}{\omega_d} \]
Percent Overshoot

- Substituting the peak time $c(t_p) = 1 - e^{-\sigma \frac{\pi}{\omega_d}} \left( \cos \pi + \frac{\sigma}{\omega_d} \sin \pi \right)$ back into the step response yields
  
  
  \[ M_p = e^{-\frac{\pi \xi}{\sqrt{1-\xi^2}}} \cdot 0 \leq \xi < 1 \]

- Notice this is only a function of the damping ratio. We can invert to find the required damping ratio for a particular overshoot

  \[ \xi = \frac{-\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}} \]
Percent Overshoot

- We can plot the relationship between percent overshoot and damping ratio.
- Two frequently used specifications are:
  - $M_p = 5\%$ for $\zeta = 0.7$
  - $M_p = 16\%$ for $\zeta = 0.5$
• The settling time can be derived in a similar manner as for a first order system

\[ e^{-\zeta \omega_n t_s} = 0.02 \]

so \[ t_s = \frac{4}{\zeta \omega_n} \]

\[ = \frac{4}{\sigma} \]
Rise Time

• Rise time is not as straightforward to compute for this case
• A precise analytical relationship does not exist
• Instead we use an approximation as suggested in Franklin (Section 3.4.1).
• Nise suggests an alternative approach based on a graph of normalised rise time

\[ t_r \approx \frac{1.8}{\omega_n} \]
Time Domain Specifications

- Lines of constant peak time, $T_p$, settling time, $T_s$, and percent overshoot, %OS and rise time
- Note: $T_{s2} < T_{s1}$;
  $T_{p2} < T_{p1}$;
  %OS$_1 < $%OS$_2$
Step response and Pole Location

- Step responses of second-order underdamped systems as poles move:
  - a. with constant real part $\sigma$;
  - b. with constant imaginary part $j\omega$;
  - c. with constant damping ratio $\zeta$
Example : Transforming Specifications

- Find allowable region in the $s$-plane for the poles of a transfer function to meet the requirements.

\[
\begin{align*}
\text{t}_r & \leq 0.6 \text{ sec} \quad \rightarrow \quad \omega_n \geq \frac{1.8}{t_r} = 3.0 \text{ rad/sec} \\
M_p & \leq 10\% \quad \rightarrow \quad \zeta \geq 0.6 \\
\text{t}_s & \leq 3 \text{ sec} \quad \rightarrow \quad \sigma \geq \frac{4}{3} = 1.5
\end{align*}
\]
Example : Transient Response through Component Design

Given the system shown here, find J and D to yield 20% overshoot and a settling time of 2 seconds for a step input torque $T(t)$. 

$T(t)$, $\theta(t)$

$K = 5 \text{ N-m/rad}$

$J$

$D$
Example: Transient Response through Component Design

- First the differential equation
  \[ J \ddot{\theta} + D \dot{\theta} + K \theta = T \]
- Find the transfer function
  \[ G(s) = \frac{1/J}{s^2 + \frac{D}{J} s + \frac{K}{J}} \]
- The second order properties are
  \[ \omega_n = \sqrt{\frac{K}{J}} \text{ and } 2\zeta\omega_n = \frac{D}{J} \]
Example: Transient Response through Component Design

• From problem statement
  \[ T_s = 2 = \frac{4}{\xi \omega_n} \]

• Hence
  \[ \frac{D}{J} = 4 \text{ and } \xi = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}} \]

• For a 20% overshoot, \( \xi = 0.456 \) so
  \[ \frac{J}{K} = 0.052 \]

• From problem statement \( K = 5 \) so \( J = 0.26 \text{kgm}^2 \) and
  \( D = 1.04 \text{Nms/rad} \)
Additional Poles and Zeros

- The preceding developing holds for a second order system with no zeros
- What happens to the system response if we add additional poles or zeros?
- This depends on their location in the s-plane
Additional Poles

• Consider an additional pole at $\alpha$

• By partial fraction expansion and completing squares we have the step response

\[
H(s) = \frac{\omega_n^2}{(s + \alpha)\left(s^2 + 2\zeta \omega_n + \omega_n^2\right)}
\]

\[
T(s) = \frac{A}{s} + \frac{B(s + \zeta \omega_n) + C \omega_d}{(s + \zeta \omega_n)^2 + \omega_d^2} + \frac{D}{s + \alpha}
\]

\[
c(t) = Au(t) + e^{-\zeta \omega_n t} \left( B \cos \omega_d t + C \sin \omega_d t \right) + De^{-\alpha t}
\]
Additional Poles

- As the pole location is moved to the left in the LHP, the effect on the response becomes less and the transient dies out more quickly.
Additional Poles

• How much further from the dominant poles does the third pole have to be for its effect on the second-order response to be negligible?
• This depends on the accuracy required by the design
• We often assume the exponential decay is negligible after five time constants
• The accuracy of this assumption should always be verified
Zeros

- Consider a zero at $\beta$
- For large $\beta$, the zero acts as a scaling factor
- For smaller $\beta$, the derivative increases the speed of response and overshoot

$$H_z(s) = \frac{(s + \beta)\omega_n^2}{(s^2 + 2\zeta\omega_n + \omega_n^2)}$$

$$= \frac{s\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} + \frac{\beta\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$$= sH(s) + \beta H(s)$$
Zeros

- If we compare the normalized step response, we find that the derivative term increases the responsiveness of the system.
• We have seen that a pole in the RHP makes a system unstable
• A zero in the RHP causes the response to be depressed
• If the derivative term is larger than the scaled response, this can lead to non-minimum phase behaviour
Effects of Pole-Zero Patterns

- For a second order system with no finite zeros, the transient response parameters are approximated by
  - Rise time: \( t_r \approx \frac{1.8}{\omega_n} \)
  - Overshoot: \( M_p \approx \begin{cases} 5\%, \xi = 0.7 \\ 16\%, \xi = 0.5 \\ 20\%, \xi = 0.45 \end{cases} \)
  - Settling Time (2%): \( t_s \approx \frac{4}{\sigma} \)
Effects of Pole-Zero Patterns

• An additional pole in the left half-plane (LHP) will increase the rise time significantly if the extra pole is within a factor of 5 of the real part of the complex poles
• A zero in the LHP will increase the overshoot if the zero is within a factor of 4 of the real part of the complex poles
• A zero in the RHP will depress the overshoot (and may cause the step response to be non-minimum phase)
Conclusions

• We have looked at the characteristics of first and second order systems
• We have derived specifications that describe the response of the system to a step input
• We have also looked at the effect of extra poles and zeros on the system response
Further Reading

• Nise
  – Sections 4.1-4.8

• Franklin & Powell
  – Section 3.3-3.5