15th Australasian Fluid Mechanics Conference
The University of Sydney, Sydney, Australia
13-17 December 2004

The Effect of Coriolis Force on Marangoni Convection

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Abstract

The effect of uniform rotation on the onset of steady and oscillatory surface-tension-driven (Marangoni) convection in a horizontal fluid layer heated from below is considered theoretically. The theoretical analysis follows the usual small-disturbance approach of perturbation theory and leads, at the marginal state, to a functional relation between the Marangoni and Taylor numbers which is then computed numerically. We present numerically a necessary and sufficient condition for oscillatory Marangoni convection to occur in a rotating fluid layer with a deformable free upper surface.

Introduction

The onset of surface-tension-gradients-driven (Marangoni) convection in a layer of fluid which is heated (or cooled) from below is a fundamental model problem for several material processing technologies, such as semiconductor crystal growth from melt, in the microgravity environment of space. As Schwabe [8] describes, typically in microgravity surface tension rather than buoyancy forces are the dominant mechanism driving the flow. In general, convection appears when a certain dimensionless parameter exceeds a critical value. This parameter is a Rayleigh number when the convection is induced by buoyancy effects due to variations in density and is a Marangoni number when surface-tension variations induce the convection.

In his pioneering work Pearson [6] showed that variation of surface tension with temperature will drive steady Marangoni convection in a fluid layer provided that the non-dimensional Marangoni number, M, (defined in the next Section) is sufficiently large and positive. Since for most fluids surface tension decreases with increasing temperature, this means that steady convection only occurs when the layer is heated sufficiently strongly from below. The most significant limitation of Pearson's [6] work was that it considered only the case of a non-deformable free surface, corresponding to the limit of strong surface tension. Scriven and Sternling [9], Smith [10] and Takashima [11] extended Pearson's [6] analysis by considering the effect of free surface deformation on the onset of steady convection and found that it dramatically destabilises the long wavelength modes. Vidal and Acrivos [14] and Takashima [12] showed numerically that oscillatory Marangoni convection is impossible when the free surface is non-deformable.

All of the work mentioned above excluded the effect of rotation of the fluid layer. The effect of rotation on Benard convection was first studied by Chandrasekhar [1]. Vidal and Acrivos [13] analysed the effect of rotation on Marangoni convection. McConaghy and Finlayson [4] re-examined Vidal and Acrivos' [13] conclusion on the possibility of oscillatory convection. Namikawa *et al.* [5] studied the case when both buoyancy and surface tension forces act together to cause the instability. Kaddame and Lebon [2, 3] investigated the onset of steady and oscillatory Benard-Marangoni convection with rotation.

In this work we use the classical linear stability theory to study the effect of rotation on the marginal curves for the onset of steady and oscillatory Marangoni convection. In particular, we

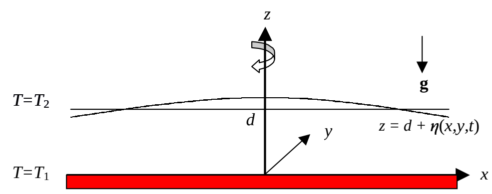

Figure 1: Sketch of the one-layer model.

show how the P_r-T_a (where the Prandtl number P_r and the Taylor number T_a are defined in the next Section) parameter space is divided into domains in which steady or oscillatory convection is preferred, and in so doing we extend the work of Kaddame and Lebon [3].

Mathematical Formulation

We wish to examine the stability of a horizontal layer of quiescent fluid of thickness d which is unbounded in the horizontal x- and y-directions. The layer is kept rotating uniformly around a vertical axis with a constant angular velocity Ω. The layer is bounded below by a thermally conducting planar boundary and above by a free surface, subject to a uniform vertical temperature gradient (see figure 1).

The fluid is Boussinesquian with a mass density ρ assumed to vary linearly on the temperature

$$\rho = \rho_0[1-\alpha_1(T-T_0)], \quad (\alpha_1 > 0), \tag{1}$$

where α_1 is the volume expansion coefficient and T_0 a reference arbitrary temperature. The variations of surface tension γ with the temperature T is assumed in the form

$$\gamma = \gamma_0 - \tau(T-T_0), \tag{2}$$

where γ_0 is a reference value of surface tension and τ is the rate of change of surface tension with the temperature. The deformation of the interface is represented by the relation

$$z = d + \eta(x,y,t), \tag{3}$$

wherein $\eta(x,y,t)$ is an *a-priori* unknown deformation with respect to the mean thickness d. In the reference state, the fluid is at rest with respect to the rotating axes and heat propagates only by conduction. When motion sets in, the velocity $\mathbf{v} = (u,v,w)$, pressure p and temperature T fields obey the usual balance equations of mass, momentum and energy (cf. Chandrasekhar [1]),

$$\nabla\cdot\mathbf{v} = 0, \tag{4}$$

$$\rho_0\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\Omega\times\mathbf{v}\right] = -\nabla p + \mu\nabla^2\mathbf{v} - \rho g\mathbf{e}_z, \tag{5}$$

$$\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T = \kappa\nabla^2 T, \tag{6}$$

where $\mathbf{g} = (0,0,-g)$ is the gravitational field, $\mathbf{e}_z = (0,0,1)$ is a unit vector in the z-direction, μ is the viscosity, κ is the thermal diffusivity and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator.

At the deformably free surface, $z = d + \eta(x,y,t)$, the boundary conditions comprise of the kinematic, the heat flux, the two shear stress and the normal stress conditions which are given by, respectively,

$$\begin{aligned}
\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} &= w, &(7)\\
k\nabla T \cdot \mathbf{n} + hT &= 0, &(8)\\
2\mu D_{\mathrm{nt}} &= \frac{\partial \gamma}{\partial T}\nabla T \cdot \mathbf{t}, &(9)\\
(p_a - p) + 2\mu D_{\mathrm{nn}} &= \gamma \nabla \cdot \mathbf{n}, &(10)
\end{aligned}$$

where h is the heat transfer coefficient, k is the thermal conductivity of the fluid, p_a is the pressure of the atmosphere, D_{ij} is the rate of strain tensor, $\mathbf{t}$ and $\mathbf{n}$ denote tangential and normal unit vectors, respectively. At the isothermal lower, rigid and plane, boundary we have the no-slip condition.

We introduce infinitesimal disturbances to the governing equations and boundary conditions by setting

$$\begin{aligned}
(u,v,w,p,T) &= (0,0,0,\bar{p},\bar{T})\\
&\quad + (u',v',w',p',\theta'), \qquad (11)
\end{aligned}$$

where the barred quantities are the basic state solutions and primed quantities represent the perturbed variables. A set of scales d, d^2/κ, ΔT is chosen for distance, time and temperature, respectively. The perturbed quantities in normal mode forms are

$$\begin{bmatrix} w' \\ \theta' \\ \zeta' \\ \eta \end{bmatrix} = \begin{bmatrix} W(z) \\ \Theta(z) \\ K(z) \\ E \end{bmatrix} \mathrm{e}^{i(a_x x + a_y y) + \sigma t}, \qquad (12)$$

where a_x and a_y are wavenumbers of disturbances in the x and y directions, respectively. W, Θ, K and E are amplitudes of vertical velocity, temperature, vertical vorticity and deflection of the free upper surface, respectively. The growth parameter σ is in general a complex variable denoted by $\sigma = \sigma_r + i\sigma_i$, where σ_r is the growth rate of the instability and σ_i is the frequency. If $\sigma_r > 0$, the disturbances grow and the system becomes unstable. If $\sigma_r < 0$, the disturbances decay and the system becomes stable. When $\sigma_r = 0$, the instability of the system, at the marginal state, sets in stationarily, provided ($\sigma_i = 0$), or oscillatorily, provided ($\sigma_i \neq 0$).

The governing equations of the perturbed state in dimensionless forms, assuming the Boussinesq approximation, are

$$\begin{aligned}
(D^2 - a^2)(D^2 - a^2 - \sigma P_r^{-1})W - T_a^* DK &= a^2 R^* \Theta, &(13)\\
(D^2 - a^2 - \sigma P_r^{-1})K &= -DW, &(14)\\
(D^2 - a^2 - \sigma)\Theta &= -W, &(15)
\end{aligned}$$

subject to

$$\begin{aligned}
W - \sigma E &= 0, &(16)\\
C_r^*[(D^2 - 3a^2 - \sigma P_r^{-1})DW - T_a^* K]&\\
- a^2(a^2 + B_o^*)E &= 0, &(17)\\
(D^2 + a^2)W + a^2 M^*(\Theta - E) &= 0, &(18)\\
D\Theta + B_i^*(\Theta - E) &= 0, &(19)\\
DK &= 0, &(20)
\end{aligned}$$

evaluated on the undisturbed position of the upper free surface $z = \pi$, and

$$W = \Theta = K = DW = 0, \qquad (21)$$

evaluated on the lower rigid boundary $z = 0$, where the operator $D = \mathrm{d}/\mathrm{d}z$ denotes differentiation with respect to the vertical coordinate z and $a = (a_x^2 + a_y^2)^{1/2}$ is the horizontal wave number of the disturbance. The starred dimensionless numbers are defined by $R^* = R/\pi^4, M^* = M/\pi^2, T_a^* = T_a/\pi^4, C_r^* = \pi C_r, B_i^* = B_i/\pi, B_o^* = B_o/\pi^2$, where the Rayleigh number, $R = \alpha g \Delta T d^3/\nu\kappa$, where ν is the kinematic viscosity, the Marangoni number, $M = \gamma \Delta T d/\rho_0 \nu \kappa$, the Taylor number, $T_a = 4\Omega^2 d^4/\nu^2$, the capillary number, $C_r = \rho_0 \nu \kappa/\gamma_0 d$, the Biot number, $B_i = hd/k$, the Bond number, $B_o = \rho g d^2/\gamma$, and the Prandtl number, $P_r = \nu/\kappa$. The Rayleigh number R accounts for buoyancy destabilising effect. The number M accounts for surface tension destabilising effect. The Taylor number T_a represents the square of the ratio between Coriolis and frictional forces. The capillary number C_r shows an idea of the rigidity of the upper free surface of the fluid layer. The Biot number B_i represents the heat flux flow through the interface, and the physical parameter Bond number B_o is the ratio between gravity effect in keeping the surface flat and the effect of surface tension in making a meniscus. The Prandtl number, P_r, stands for the ratio between thermal and heat diffusivities.

Solution of the Linearised Problem

Combining equations (13)–(15) then gives a single linear eighth-order ordinary differential equation for Θ,

$$\begin{aligned}
&(D^2 - a^2 - \sigma)\left[(D^2 - a^2)(D^2 - a^2 - \sigma P_r^{-1})^2 + T_a^* D^2\right]\Theta\\
&+ a^2 R^*(D^2 - a^2 - \sigma P_r^{-1})\Theta = 0. \qquad (22)
\end{aligned}$$

Equation (22) together with the boundary conditions (16)–(21) constitute a linear eigenvalue problem for the unknown temporal exponent σ. Relation (17) gives the expression for the surface deflection E in terms of the other quantities. In the general case $\sigma \neq 0$ we seek solutions in the forms

$$W(z) = AC\mathrm{e}^{\xi z}, \quad K(z) = BC\mathrm{e}^{\xi z}, \quad \Theta(z) = C\mathrm{e}^{\xi z}, \qquad (23)$$

where the complex quantities A, B and C and the exponent ξ are to be determined. Substituting these forms into the equations (13)–(15) and eliminating A, B and C we obtain an eighth-order algebraic equation for ξ, namely

$$\begin{aligned}
&(\xi^2 - a^2 - \sigma)\left[(\xi^2 - a^2)(\xi^2 - a^2 - \sigma P_r^{-1})^2 + T_a^* \xi^2\right]\\
&+ a^2 R^*(\xi^2 - a^2 - \sigma P_r^{-1}) = 0, \qquad (24)
\end{aligned}$$

with eight distinct roots $\xi_1, \ldots, \xi_8$. Denoting the values of A, B and C corresponding to ξ_i for $j = 1, \ldots, 8$ by A_j, B_j and C_j we can use equations (14) and (15) to determine A_j and B_j to be

$$A_j = -(\xi_j^2 - a^2 - \sigma), \quad B_j = -\frac{\xi_j A_j}{\xi_j^2 - a^2 - \sigma P_r^{-1}}, \qquad (25)$$

for $j = 1, \ldots, 8$. The boundary conditions (16)–(21) can be used to determine the eight unknowns $C_1, \ldots, C_8$ (up to an arbitrary multiplier), and the general solution to the stability problem is therefore

$$W(z) = \sum_{j=1}^{8} A_j C_j \mathrm{e}^{\xi_j z}, \quad K(z) = \sum_{j=1}^{8} B_j C_j \mathrm{e}^{\xi_j z}, \quad \Theta(z) = \sum_{j=1}^{8} C_j \mathrm{e}^{\xi_j z}. \qquad (26)$$

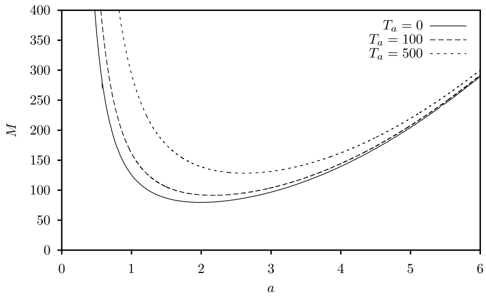

Figure 2: Numerically-calculated marginal curves for the onset of steady Marangoni convection plotted as functions of a in the case $C_r = 0$ and $B_i = 0$ for several values of T_a.

Imposing boundary conditions (16)–(21) yields a linear system $P\mathbf{A} = \mathbf{0}$, where $\mathbf{A} = [A_1, \ldots, A_8]^T$. In general, the 8×8 coefficient matrix P (whose entries depend on a, M, R, σ, C_r, T_a, P_r, B_o and B_i) is complex and may be rather complicated, and so, in general, it has to be calculated either numerically or symbolically using a symbolic algebra package. In this work we use both approaches. We use a FORTRAN 77 program employing the Numerical Algorithms Group (NAG) routine F03ADF to evaluate the determinant of P using LU factorisation with partial pivoting. A modification of Powell's [7] hybrid algorithm, which is a combination of Newton's method and the method of steepest descent, implemented using NAG routine C05NBF is then used to find the eigenvalues of P by solving the two nonlinear equations obtained from the real and imaginary parts of the determinant of P.

Marginal Stability Curves

In this work we shall concentrate on the problem of the onset of steady and oscillatory Marangoni convection, i.e. we set $R = 0$. The marginal stability curves in the (a, M) plane on which $\sigma_r = 0$ separate regions of unstable modes with $\sigma_r > 0$ from those of stable modes with $\sigma_r < 0$. The critical Marangoni number for the onset of convection is the global minimum of M over $a \geq 0$.

For steady convection ($\sigma = 0$), the dispersion relation $F(a, M, C_r, T_a, B_o, B_i) = 0$ takes the linear form $D_1 + MD_2 = 0$, where D_1 and D_2 are two 6×6 determinants which depend on the whole set of parameters of the problem except M. Given any set of values for R, T_a, C_r, B_i, B_o, we can determine the Marangoni number as a function of the wave number a.

Figure 2 shows typical marginal stability curves for the onset of steady Marangoni convection for various values of the Taylor number T_a in the case when the free surface is undeformable, $C_r = 0$, and insulating, $B_i = 0$. In this case the problem is independent of B_o. As a validation of our algorithm we found that as $T_a \to 0$ the marginal curves tend to that obtained by Pearson [6] for the pure Marangoni problem without rotation, $T_a = 0$. Figure 2 clearly shows that in the cases investigated the effect of rotation is to stabilise the layer. Rotation introduces vorticity into the fluid which then causes the fluid to move in the horizontal planes with higher velocity. The velocity of the fluid perpendicular to the planes reduces, thus the onset of convection is inhibited (Chandrasekhar [1]).

While in practice the value of C_r may be very small (for a 1 cm layer of water open to air at 20^o C we have $C_r \sim 10^{-7}$) it will inevitably be non-zero. Figure 3 shows typical marginal stability curves for the onset of steady Marangoni convection for a range of values of T_a for $C_r = 0.001$, $B_o = 0.1$ and $B_i = 0$. As shown in figure 3 the marginal curves can have a local minimum value at $a = 0$. There exists a critical Taylor number, say T_{ac} below which the onset of convection is at $a = 0$ and above which the onset of convection is at $a = O(1)$. More interesting is the case when $T_a = T_{ac}$ in which a competition between two different modes is possible. If the free surface is allowed to deform ($C_r \neq 0$) then the marginal stability curves differ fundamentally from those in the case $C_r = 0$ in the region $a \ll 1$ and depend critically on T_a as depicted in figure 4. In the cases investigated in figure 4 variations in C_r has a minute effect on the marginal curves as a gets bigger.

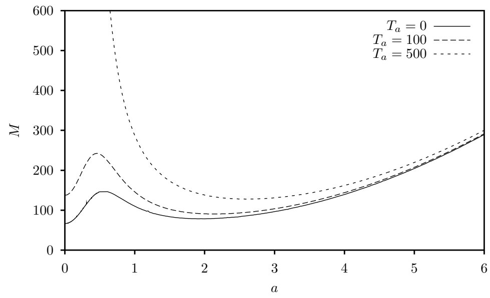

Figure 3: Numerically-calculated steady marginal curves for the onset of Marangoni convection plotted as functions of a in the case $C_r = 0.001$, $B_o = 0.1$ and $B_i = 0$ for several values of T_a.

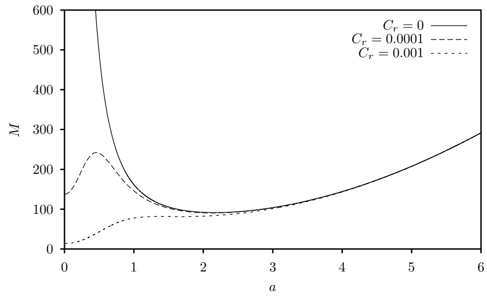

Figure 4: Numerically-calculated steady marginal curves for the onset of Marangoni convection plotted as functions of a in the case $T_a = 100$, $B_o = 0.1$ and $B_i = 0$ for several values of C_r.

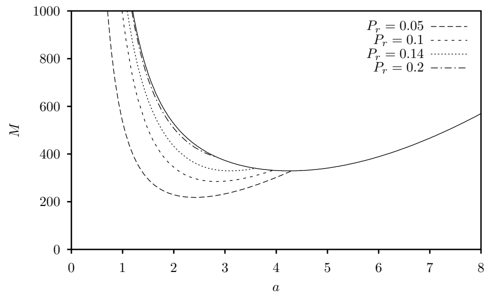

Figure 5: Numerically-calculated steady marginal curves for the onset of steady (solid) and oscillatory (dashed) Marangoni convection plotted as functions of a in the case $C_r = 0$, $T_a = 5000$, $B_o = 0.1$ and $B_i = 0$ for several values of P_r.

Kaddame and Lebon [3] showed that convection can set in as oscillatory ($\sigma \neq 0$) motions for the case when the free surface is flat, but no complete marginal curves were given. In figure 5 we plot both steady and oscillatory marginal curves in the case $T_a = 5000$, $C_r = 0$, $B_o = 0.1$ and $B_i = 0$ for several values of P_r. There exists a certain critical value of $P_r = P_{rc}$ (depending on the other problem parameters) below which the onset of convection is oscillatory. In figure 6 we plot P_{rc} as a function of T_a in the case $B_o = 0.1$ and $B_i = 0$ for several values of C_r. Each curve in figure 6 defines the boundary between the steady and oscillatory domains. Points below each curve in figure 6 rep-

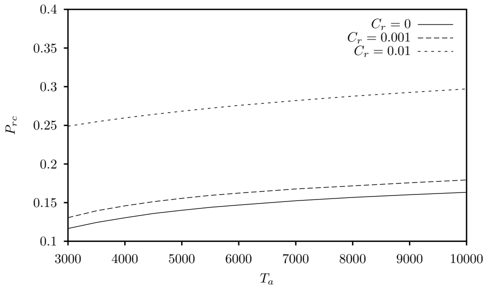

Figure 6: Numerically-calculated critical values of P_r below which oscillatory convection is prefered plotted as functions of T_a in the case $B_o = 0.1$ and $B_i = 0$ for several values of C_r.

resent parameter combinations (T_a, P_{rc}) for which convection sets in as oscillatory motions, while points above each curve are those for which oscillatory convection is preferred. In the cases investigated, for a fixed value of T_a, increasing C_r has the effect of increasing P_{rc}.

Conclusions

In this work we used classical linear stability theory to investigate the effect of rotation on the onset of steady and oscillatory Marangoni convection in a horizontal planar layer of fluid heated from below. The results showed the stabilising effect of the rotation and the possibility of the co-existence of two different modes at the onset of convection. In particular, we showed how the P_r-T_a parameter space is divided into domains in which steady or oscillatory convection is preferred.

Acknowledgments

The authors acknowledge the financial support from Malaysian Ministry of Science, Technology and Environment through IRPA grant no. 09-02-02-0078-EA206.

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