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The stability of decaying pipe flow

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Abstract

We consider the decay and subsequent instability of the fully developed flow within a pipe of circular cross section when the pipe is suddenly closed.

Introduction

The behaviour of the flow within a suddenly blocked pipe has important applications across a wide range of disciplines. Two such examples are the so called water-hammer effect which occurs when a valve is suddenly closed in a pipe and the rhythmic opening and closure of the *aortic valve* and the *pulmonic valve* in the heart during ventricular ejection. In both applications an unsteady flow develops which typically exhibits a transient turbulent state (see, for example, Refs. [3] and [13]).

It was the physiological applications that led Wienbaum & Parker [16] to first consider the problem of the decay of the flow in a suddenly blocked channel or pipe. They gave the problem its correct mathematical formulation and employed an approximate technique based upon the Pohlhausen method to describe the flow. This work allowed them to demonstrate that the decaying channel flow develops points of inflection thus suggesting that the flow would be susceptible to wave-like instabilities. The stability of the flow was subsequently considered by Hall & Parker [7] who employed a WKBJ style approximation, based upon the assumption of large flow Reynolds number, to derive a quasi-steady Orr-Sommerfeld equation describing the flow stability.

The theoretical result that the decelerating flow in a suddenly blocked pipe is unstable to wave-like disturbances is in qualitative agreement with *in vivo* measurements of turbulence levels in the ascending aorta (for example Ref. [13]). Hall & Parker [7] demonstrated that the decaying flow within a suddenly blocked channel is unstable, due to the inflectional nature of the stream-wise velocity profiles, for Reynolds numbers as low as $O(10^2)$. Some care must be taken in interpreting these results, since the quasi-steady approximation they employed requires the flow Reynolds number to be simultaneously large (for the asymptotic approximation to be valid) and finite (to justify retaining viscous terms in the resulting Orr-Sommerfeld equation).

One of the drivers of the renewed interest in the behaviour of the flow in a suddenly blocked pipe occurs in the water industry where considerable attention has been given to the problem of detecting leaks in pipeline systems using inverse transient techniques, see Ref. [10]. For this technique to be fully implemented it is necessary to be able to differentiate between damping due to leaks and damping due to unsteady friction resulting from the (possibly) turbulent flow within the pipeline. Current models for the unsteady friction within pipes, such as that of Vardy & Brown [15], are largely empirical and typically underpredict the amplitude and the phase of the pressure response within the pipeline. Recent work by Lambert *et al.* [9] suggests that this lack of agreement between theory and experiment may be largely due to the empirical nature of the friction models used. Their results highlight the need for an improved understanding of the flow within the unsteady boundary layer, both laminar and turbulent.

It is the aim of this paper to quantify the stability properties of the decaying flow in a suddenly blocked pipe.

Formulation

Consider the pressure driven flow within a cylindrical pipe of non-dimensional radius $r = 1$ which is suddenly blocked at $x = 0$ at time $t = 0$, where (r,θ,x) are the usual cylindrical polar coordinates and (v,w,u) is the corresponding velocity field. Prior to blockage the flow is assumed to be fully developed, laminar, Poiseuille flow with axial velocity given by $u(r,x) = 1-r^2$.

As noted by Wienbaum & Parker [16], the pressure wave which results from the sudden valve closure acts to freeze the vorticity within the flow in the state which existed prior to the closure. Thus, provided the Mach number of the flow is small (which it invariably is for most pipeline applications), immediately after the passage of the pressure wave the vorticity distribution within the flow is the same as it was before the closure. Solving the vorticity equation demonstrates that immediately after the blockage, roughly two pipe radii downstream of the blockage, the flow is uni-directional and given by

$$u_0(r,x) = \frac{1}{2} - r^2 \qquad (t=0). \tag{1}$$

Following the blockage, the flow develops on the viscous diffusion time-scale $\tau = Re^{-1}t = O(1)$, where Re is the flow Reynolds number.

Assuming then that the flow within the blocked pipe is now uni-directional, we write $(v,w,u) = (0,0,U_0(r,\tau))$. Substitution of this expression into the full Navier-Stokes equations yields

$$\frac{\partial U_0}{\partial \tau} = \frac{\partial^2 U_0}{\partial r^2} + \frac{1}{r}\frac{\partial U_0}{\partial r} + \phi'(\tau), \tag{2a}$$

which must be solved subject to

$$U_0(r,0) = \frac{1}{2} - r^2, \quad 0<r<1, \tag{2b}$$

$$U_0(1,\tau) = 0 \quad \tau > 0. \tag{2c}$$

In (2a) the term $\phi'(\tau)$ denotes the unsteady axial pressure gradient; this must be determined as part of the solution process. In order to do this we impose the condition that the integrated volume flux across any cross-section must vanish. Thus

$$\int_0^1 rU_0(r,\tau)\mathrm{d}r = 0. \tag{2d}$$

The system (2) can be solved be solved in two ways. The first involves taking the Laplace transform in τ, solving the resulting ordinary differential equation and then taking the inverse Laplace transform to obtain the velocity field $U_0(r,\tau)$. Due

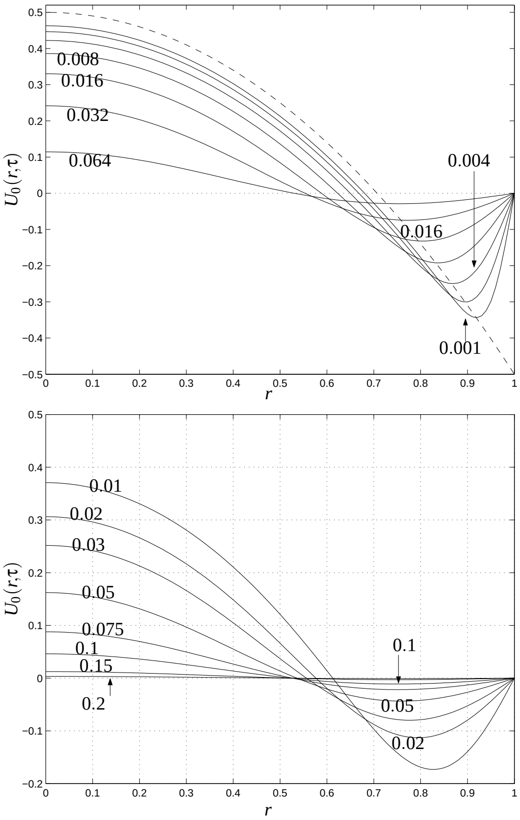

Figure 1: Plot of streamwise velocity $U_0(r,\tau)$ versus r for times τ as labelled.

to its complexity the inverse Laplace transform must be calculated numerically. The second approach involves discretizing the equation (2a), in both space and time, and solving the discretized system numerically. It is readily shown (see Refs. [8], [16]) that the streamwise pressure gradient term has a singularity at time $t = 0$ in the form $\phi'(\tau) = O(\tau^{-1/2})$ as $\tau \to 0$ (this is a simple consequence of the impulsive nature of the blockage). For this reason a simple marching-in-time scheme, with initial conditions imposed at $\tau = 0$, is not suitable for solving (2). To use such a scheme an accurate small-time solution must be developed. Details of this can be found in Jewell & Denier [8]. For our purposes it is sufficient to note that, for small τ, we can write

$$\phi(\tau) = -\frac{2}{\sqrt{\pi}}\tau^{1/2} + \frac{5}{2}\tau + O(\tau^{3/2});$$

the velocity field $U_0(r,\tau)$ $(0 < \tau \ll 1)$ can be written in the form of an infinite power series in powers of $\tau^{1/2}$ (details can be found in Ref. [8]).

This small τ solution was taken as the starting-point for a Crank-Nicolson finite-difference marching scheme as follows. Given $U_0(r,\tau)$ and $\phi'(\tau)$ at time τ_0 we use the value of $\phi'(\tau_0)$ as a guess for the value of $\phi'(\tau_1)$ (where $\tau_1 = \tau_0 + \Delta\tau$). The inhomogeneous discretized equations are then solved, subject to the boundary condition (2b) to give $\tilde{U}_0(r,\tau_1)$; this will be the "correct" value if and only if the flux condition (2d) is satisfied. In general this will not be the case thus allowing us to set-up an iteration scheme, based upon the flux condition, which can be used to update $\phi'(\tau_1)$. We chose to employ Newton iteration for this task.

The results of our calculations are presented in figure 1 which show the decay of the streamwise velocity. For small times τ we observe the rapid change as the flow adjusts from the inviscid slip condition to the no-slip boundary condition on $r = 1$.

Linear stability of the flow

To consider the stability of the flow we proceed in two ways. Firstly we undertake a a classical linear stability analysis which invokes the *quasi-steady approximation* that the basic flow does not vary significantly over the $O(t)$ time scale characteristic of Orr-Sommerfeld modes. We then relax the quasi-steady assumption thus taking into account the $O(\tau)$ evolution of the basic flow and focus our attention on *transient pseudomodes*, that is, flow perturbations which are capable of significant transient growth.

Normal mode analysis

Here we look for flow perturbations in the form

$$\begin{aligned}(\mathbf{u}, p) &= (0, 0, U_0(r,\tau), p) \\ &+ \varepsilon(U_r(r), U_\theta(r), U_x(r), P(r))\, e^{i[\alpha(x-ct)+k\theta]}, \quad (3)\end{aligned}$$

where U_x, U_r and U_θ are the axial, radial and circumferential velocity components respectively, and c denotes a complex-valued wave-speed. The diffusion time scale τ is treated as a parameter (the quasi-steady approximation), along with the Reynolds number Re, the axial wavenumber α and the azimuthal wavenumber k $(k = 0, 1, 2, \ldots)$. The case $k = 0$ corresponds to two-dimensional perturbations, that is $U_\theta = 0$. The governing equations for $(\mathbf{U}(r), P(r))$ are

$$i\alpha(U_0 - c)U_x = -i\alpha P - U_0' U_r + Re^{-1}\mathcal{L}U_x, \quad (4a)$$

$$i\alpha(U_0 - c)U_r = -P' + Re^{-1}\left[\mathcal{L}U_r - r^{-2}U_r - 2ikr^{-2}U_\theta\right], \quad (4b)$$

$$i\alpha(U_0 - c)U_\theta = -ikr^{-1}P + Re^{-1}\left[\mathcal{L}U_\theta - r^{-2}U_\theta + 2ikr^{-2}U_r\right], \quad (4c)$$

$$0 = i\alpha U_x + \left(\frac{\partial U_r}{\partial r} + r^{-1}U_r\right) + ikr^{-1}U_\theta. \quad (4d)$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - (\alpha^2 + k^2 r^{-2}).$$

The full system (4a)–(4d) is to be solved subject to the no-slip condition $\mathbf{U}(1) = \mathbf{0}$. Boundary conditions at $r = 0$ (adapted from [6]) are

$$k = 0: \quad U_r = 0, \quad U_x' = 0, \quad P' = 0. \quad (4e)$$

$$k = 1: \quad U_x = 0, \quad P = 0, \quad U_r' = 0, \quad U_c = i\alpha U_r. \quad (4f)$$

$$k > 1: \quad \mathbf{U} = \mathbf{0}, \quad P = 0. \quad (4g)$$

When discretized, system (4) reduces to a generalized eigenvalue problem for the complex wavespeed

$$\mathbf{A}(\mathbf{U}, \mathbf{P})^T = c\mathbf{B}(\mathbf{U}, \mathbf{P})^T. \quad (5)$$

Discretization was performed using an order-N Chebyshev pseudospectral scheme, yielding an eigenvalue problem of approximate size $3N$ or $4N$. The order $N_{\min}$ required to resolve the first eigenvalue to five decimal places was found to range from fifteen to twenty five, depending on choice of parameters (provided that $\tau \gtrsim 0.001$). Furthermore, resolution of the ten leading eigenvalues was generally possible at $N = N_{\min} + 10$.

The results from our calculations are given in figure 2. We note that the flow is unconditionally stable to axisymmetric modes

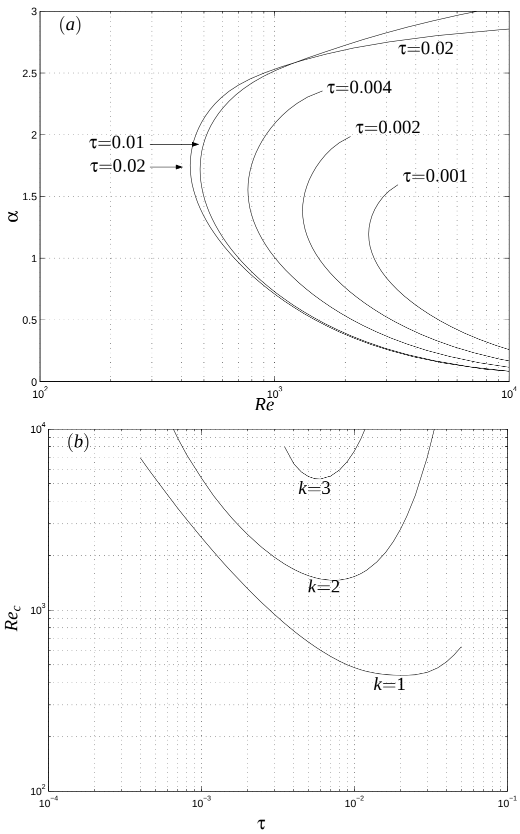

Figure 2: Plots of (a) Neutral-stability curves for blocked-pipe flow and (b) the corresponding critical Reynolds number for azimuthal wavenumbers $k = 1, 2, 3$. The solid line corresponds to the neutral-stability curves.

$(k = 0)$. Neutral curves, for $k = 1$, are presented in figure 2a. From this figure we clearly observe the variation of the critical Reynolds number with time τ. In figure 2b we plot the critical Reynolds number versus τ for $k = 1, 2, 3$. The most unstable mode corresponds to $k = 1$ and has a minimum critical Reynolds number of $Re_c \approx 440$ which occurs at a time $\tau = 0.02$. These results are in qualitative agreement with those of Ref. [5].

It has long been known that eigenmodes predict long-term rather than short-term behaviour. Whether this is a reliable guide to short-term behaviour depends on whether the eigenmodes are orthogonal and non-degenerate: where this is not the case, transient growth may be possible even if all individual eigenmodes decay and we now turn our attention to this problem.

Transient growth analysis

It is only in recent years that it has been recognised that the eigenmodes of some flows are not orthogonal. This is indeed the case for Couette and plane-Poiseuille flows. For these flows it is possible to describe a linear combination of eigenmodes which interfere destructively in the early stages, before separating out to produce significant transient growth in the intermediate stage (see Refs. [2], [12] and [14]). Eventually, in a purely linear model, this linear combination decays exponentially in accordance with classical predictions. This leads to the conjecture that in practice the transient may attain a critical amplitude beyond which non-linear effects destabilize the flow. This is supported by the numerical results presented in Refs. [1], [4] and [14].

There are two basic numerical techniques of linear transient analysis which are presented Refs. [2], [12] and [14]. We focus our attention on determining the *explicit transient pseudomodes*. As in classical stability analysis, the pseudomode is considered to be a flow perturbation arising instantaneously at some time $t = 0$. Typically, it attains some amplification factor $g = O(Re)$, that is $g_{\max} \approx (Re/Re_0)$ where $Re_0 \ll Re_c$, before decaying exponentially as $t \to \infty$.

The present problem differs in one important respect from Couette and Poiseuille flows: the basic flow for a blocked pipe is unsteady. To the extent that the quasi-steady assumption is valid, the explicit-transient technique is directly applicable. To this end, we denote the transient response by

$$\tilde{\mathbf{u}}(t) \equiv \tilde{\mathbf{u}}(t, r, x, \theta; \tau_0, Re, \alpha), \tag{6}$$

where $t = 0$ and $\tau = \tau_0$ denote the time of commencement. Let this transient be approximated by the J leading eigenmodes of the basic flow at time τ_0:

$$\tilde{\mathbf{u}}(t) = \sum_{j=1}^{J} \gamma_j \tilde{\mathbf{u}}_j(t; \tau_0, Re, \alpha). \tag{7}$$

Following Trefethen *et al.* [14] we define the transient-growth factor via its energy norm over space as

$$g(t) = \frac{\|\tilde{\mathbf{u}}(t)\|}{\|\tilde{\mathbf{u}}_0\|} = \left(\frac{\gamma^* \mathbf{R}(t)\gamma}{\gamma^* \mathbf{R}_0 \gamma}\right)^{\frac{1}{2}} \tag{8}$$

where $R_{jk}(t) = \langle \tilde{\mathbf{u}}_j(t), \tilde{\mathbf{u}}_k(t)\rangle$. In order to relax the quasi-steady assumption we must track the temporal and spatial evolution of each of the original eigenmodes:

$$\tilde{\mathbf{u}}_j(t) = \exp\left[-\int_0^t \tilde{\mathbf{A}}(\tau_0 + Re^{-1}t')\,\mathrm{d}t'\right] \tilde{\mathbf{u}}_j(0) \tag{9}$$

where $\tilde{\mathbf{A}}$ is the matrix evolution operator defined in Jewell & Denier [8]. The matrix-exponential was evaluated using the Matlab function `expm`, together with the formula

$$\tilde{\mathbf{u}}_j(t) = \exp\left[-t\tilde{\mathbf{A}}(\overline{\tau})\right] \tilde{\mathbf{u}}_j(0), \tag{10}$$

where $\tilde{\mathbf{A}}(\overline{\tau})$ is understood to mean $\tilde{\mathbf{A}}$ is evaluated using the mean basic flow for $\tau_0 < \tau < (\tau_0 + Re^{-1}t)$.

We calculate the maximum possible growth $g^*(t)$ at some time $t = t_1$, obtaining what Trefethen *et al.* [14] have termed the Butler-Farrell optimum transient at $t = t_1$. Obtained by setting $\partial g/\partial\gamma = \mathbf{0}$ at $t = t_1$, the Butler-Farrell transient corresponds to the dominant eigenmode of the generalized eigenvalue problem

$$\mathbf{R}(t_1)\gamma = \lambda \mathbf{R}_0 \gamma, \qquad \text{with} \qquad g^*(t_1) = \sqrt{\lambda_1(t_1)}. \tag{11}$$

Results are shown in Figures 3 and 4. Figure 3 shows the function $g^*(t)$ for a variety of parameter values (Re, τ_0) and figure 4 shows $g_{\max} \equiv \max\{g^*(t)\}$ as a function of Reynolds number. It is found that

$$g_{\max} \approx (Re/Re_0) \quad \text{where} \quad Re_0 \approx 120 \qquad (Re < 1000). \tag{12}$$

The quasi-steady estimate $g_{\max}(Re, \tau_0)$ proves to be reasonably accurate (provided that $Re < Re_c(\tau_0)$, where $Re_c(\tau_0)$ is the classical critical value). This is due to the weak dependence of $g_{\max}(Re, \tau_0)$ on τ_0; it does not demonstrate that the quasi-steady approximation is intrinsically valid. On the contrary, figure 3c shows that a given transient persists almost as long as the basic flow itself, continuing to grow until $\tau \approx 0.05$; at this time the

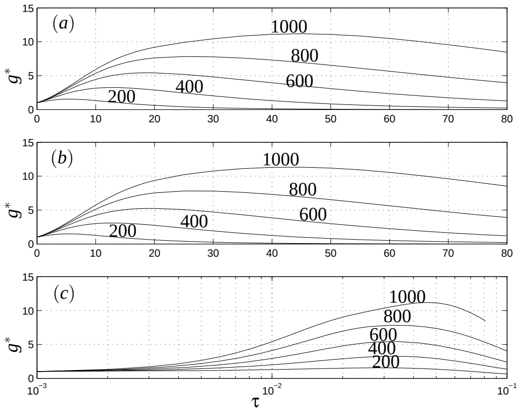

Figure 3: Butler-Farrell transient-growth profile $g^*(t_1)$ for $\alpha = 1.5$. The transient commences at (a) $\tau_0 = 0.001$ and (b) $\tau_0 = 0.004$ and (c) as for (a) but plotted against $\tau_1 = \tau_0 + Re^{-1} t_1$

viscous boundary-layer, which has grown from the pipe-wall almost fills the whole pipe (see Jewell & Denier [8] for details).

These results suggest that the precise classical value $Re_c(\tau)$ is of limited physical significance. Below this limit, transient growth may be possible. Above it, transition to turbulence may not be realized – the basic flow may decay too soon to permit much growth of the dominant eigenmode.

To explore this idea further, we compare our results with those of Trefethen *et al.* [14] for Couette and plane-Poiseuille flows (Figure 4). Couette flow (for which $Re_0 \approx 29$) is known to transition to turbulence in the range $350 < Re < 3500$; Poiseuille flow ($Re_0 \approx 71$) becomes turbulent for $10^3 < Re < 10^4$. On this basis we conjecture that transition-to-turbulence is usually associated with transient growth factors in the range 15 to 50. For the case of a blocked pipe, this corresponds to $1100 \lesssim Re \lesssim 1800$.

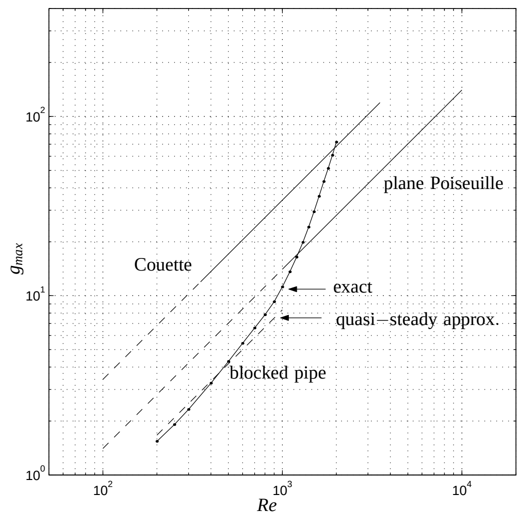

Figure 4: Maximum transient growth $g_{\max} \equiv \max\{g^*\}$. Corresponding results from Couette and plane-Poiseuille flows are provided for comparison (solid and dashed sections correspond to transitional and unstable flow, respectively).

Conclusions

We have demonstrated that the flow within a suddenly blocked pipe is unstable to non-axisymmetric disturbances. The flow also supports modes which undergo transient growth. Comparison with Couette and plane Poiseuille flow suggests that these transients may play an important role in the transition process.

Acknowledgements

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