15th Australasian Fluid Mechanics Conference
The University of Sydney, Sydney, Australia
13-17 December 2004

A steady 'filling box' solution with zero net buoyancy flux

G. O. Hughes, R. W. Griffiths and J. C. Mullarney

Research School of Earth Sciences
The Australian National University, Canberra ACT 0200, AUSTRALIA

Abstract

We present a simple flow model and solution to describe 'horizontal convection' driven by a gradient of temperature or heat flux along one horizontal boundary. Following laboratory observations for the steady-state convection, the model is based on a localized turbulent plume that maintains a stably stratified interior. In contrast to the 'filling box' process of Baines and Turner [1], the convective circulation involves vertical diffusion in the interior and a stabilising buoyancy flux distributed over the horizontal boundary. The latter flux causes the density distribution to reach a steady state. Applied to the laboratory experiments of Mullarney *et al*. [5], the solution successfully predicts the top-to-bottom density difference and the thickness of the boundary layer adjacent to the horizontal boundary. We conclude that the turbulent 'recycling box' model is a useful way to describe this form of convection.

Introduction

Horizontal convection driven by a horizontal gradient of temperature or heat flux along the top or bottom boundary of a volume of fluid has been mooted as a model for the global overturning circulation of the oceans in response to net heating at low latitudes and a net cooling at high latitudes [11]. In this form of convection, heat enters over a section of the boundary (such as the ocean surface) and is removed at an equal rate elsewhere on that boundary. The resulting boundary layer flow is observed to feed into a plume that penetrates through the depth of the box, at one end of the box [9, 10, 5, see figure 1]. For sufficiently large Rayleigh numbers the plume is turbulent [5].

In a companion paper [2] we have examined the implications of potential energy generation by the buoyancy fluxes at the horizontal boundary and of the rates of dissipation of energy in the box interior. We have shown that the steady-state buoyancy-driven flow requires a level of turbulence sufficient to mix heat into the interior at a rate equal to the cooling flux delivered by the turbulent plume. On energy grounds alone it is possible to predict the depth of the boundary layer adjacent to the surface at which the heating and cooling is applied, and to estimate the magnitude of the interior mixing rate (the vertical diffusivity). However, in order to predict the temperature (density) distribution and the mass flux (given an imposed heat flux), it is necessary to construct an explicit solution for the flow. This is the purpose of the current paper.

The 'recycling box' model

Horizontal convection has much in common with 'filling box' flows [1], where a localised destabilising source of buoyancy maintains a turbulent plume and drives a circulation in the box. Fluid in the plume penetrates rapidly through the depth of the box, whereupon it becomes part of a broad gradual return flow in the box interior towards the level of the plume source. The density anomaly of fluid in the plume is reduced with distance from the source by entrainment and mixing of water from the box interior into the plume. Consequently a density stratification is established throughout the depth of the box. As in the case of horizontal convection, the filling box stratification is strongest at levels near the plume source (in the 'thermocline') and relatively weak throughout the remainder of the box interior. The analysis of Baines & Turner [1] shows that the filling box flow reaches an asymptotic steady state in which the density gradient and velocity field is constant. Crucially, however, the density field continues to evolve since the filling box flow is driven by a continuous net input of buoyancy.

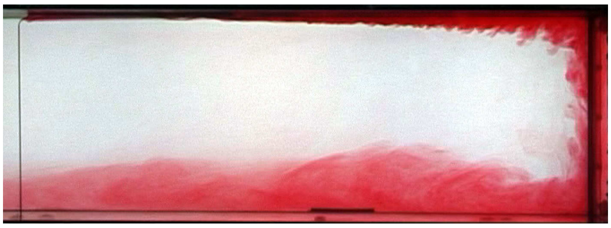

Figure 1: Dye visualisation of the convective flow driven in a box by differential heating along the lid [5]. In this experiment surface heat fluxes q_h and q_c $(= -q_h)$ per unit box width are applied over the left- and right-hand halves of the lid, respectively. The field of view shows only the right-hand (cooled) end of the box. The thin vertical line at the left of the photograph corresponds to the centre of box, where dye was introduced into the boundary layer adjacent to the lid. Heat is transferred to/from this boundary layer, maintaining a horizontal density gradient and a flow directed from the heated region to the cooled region (left to right). A turbulent outflow from the plume is visible at the base of the box, directed from right to left. For sufficiently large heat fluxes most of the temperature variation occurs across the lid boundary layer, which occupies a small fraction of the total depth.

Here, we develop a steady filling box (which we term a 'recycling box') model to describe horizontal convection. We ensure that the density field in the box is in a steady state by applying a stabilising buoyancy source at the same horizontal boundary as the destabilising buoyancy source driving the plume. The stabilising source is of such a strength that the net input of buoyancy into the box is zero. Vertical diffusion in the box interior is then important in maintaining the density distribution.

Without loss of generality, we consider in the remainder of this paper horizontal convection driven by fluxes applied at the upper surface. The destabilising buoyancy source then corresponds to cooling and the stabilising flux to heating, as shown in figures 1 and 2. Although the destabilising and stabilising fluxes were applied symmetrically over a large area either side of the tank centre in figure 1, the resulting flow is highly asymmetric with a tightly confined plume. The global overturning circulation in the oceans displays a similar asymmetry [11]. These observations suggest a useful approximation: we treat the cooling flux at the surface as highly localised, and distribute the heat input uniformly over the remainder of the surface. The exact position of the cooling and the size of the cooling area are assumed not to play an important role in determining the vertical structure of the flow, provided there is no net heat input. The rate of heat loss q_c per unit box width at the surface produces a buoyancy flux

$$F_c = \frac{\alpha g q_c}{c_p} \tag{1}$$

per unit box width, where α is the expansion coefficient, g is the gravitational acceleration and c_p is the specific heat capacity of water at constant temperature. The plume is assumed to be turbulent, as in the filling box [1, 4, 8]. Experiments [5] confirm this to be the case (figure 1) at Reynolds and Rayleigh numbers characterising laboratory-scale flows.

We consider here an isolated vertical line plume in a rectangular box (figure 2), and note that our model is equally applicable to a 'half-plume' against the boundary (as in figure 1). The ratio of height to length of the box is assumed to be small so that horizontal velocities are much greater than vertical velocities.

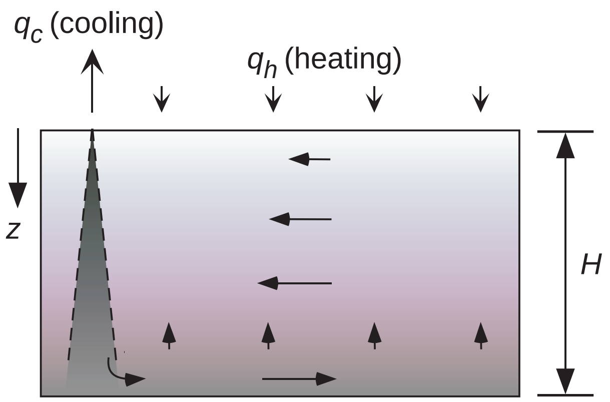

Figure 2: Schematic diagram of the 'recycling box' model for horizontal convection. Localised surface cooling at a rate q_c per unit box width leads to a tightly confined region of downwelling (represented as an isolated vertical line plume) from the surface to the bottom in a box of depth H and length $2L$. By symmetry, a 'half-plume' (against the vertical boundary) may be modelled by defining the box length to be L. Heating at a rate $q_h = -q_c$ per unit box width is distributed over the surface allowing the flow to reach a steady state. If the sinking is turbulent, entrainment drives recirculation at depth.

Plume model

We assume here that the plume width is much smaller than the length of the box. For simplicity we shall also assume that the plume does not interact with the confining (vertical) boundary of the box, but note that by symmetry the model can be applied to a 'half-plume' (figure 1) that sinks against the vertical boundary. Following Baines & Turner [1], we assume that profiles of mean velocity $\overline{W}$ and density anomaly $\overline{\rho} - \rho_e$ through the plume cross-section are similar at all depths and well approximated by a Gaussian form, i.e.

$$\overline{W}(x,z) = W_p(z)\exp\left[-\frac{x^2}{R^2(z)}\right], \tag{2}$$

and

$$\overline{\rho}(x,z) - \rho_e(z) = \left[\rho_p(z) - \rho_e(z)\right]\exp\left[-\frac{x^2}{R^2(z)}\right], \tag{3}$$

where x is the distance from the plume axis and $\overline{\rho} - \rho_e$ is the density anomaly in the plume relative to water in the interior at the same depth but far removed from the plume. We define the vertical coordinate z increasing downwards with the plume source at the origin. We have defined R to be the 1-σ plume width and $\rho_p - \rho_e$ and W_p to be the mean density anomaly and mean vertical velocity on the plume axis, respectively. Upon integrating in the horizontal plane, the equations describing conservation of volume, momentum and buoyancy in the plume can be written (cf. [1])

$$\frac{d}{dz}\left[\sqrt{\pi} R W_p\right] = 2U_e = 2EW_p, \tag{4}$$

$$\frac{d}{dz}\left[\frac{RW_p^2}{2\sqrt{2}}\right] = R\frac{g\left(\rho_p - \rho_e\right)}{\rho_r}, \tag{5}$$

and

$$\frac{d}{dz}\left[\frac{RW_p g\left(\rho_p - \rho_e\right)}{\sqrt{2}\rho_r}\right] = RW_p\frac{d}{dz}\left[\frac{g\left(\rho_r - \rho_e\right)}{\rho_r}\right], \tag{6}$$

where ρ_r is a reference density and E is a constant characterising the ratio of an entrainment velocity U_e to the vertical velocity W_p in the plume. The entrainment velocity can be interpreted as the rate of increase with z of the plume volume flux per unit box width, and E takes a value of approximately 0.1 for a Gaussian plume [1, 12].

Interior model

As in the filling box model of Baines & Turner [1], conservation of volume in the box interior is expressed as

$$\frac{\sqrt{\pi}}{2}RW_p = -LW_e, \tag{7}$$

where W_e is the mean vertical velocity in the interior over the cross-sectional area of the box. Equation (7) holds if the box length is defined to be $2L$ for isolated plumes and to be L for 'half-plumes' against a vertical boundary in the box.

The density field ρ_e in the box interior must vary with depth (cf. [1]). However, in contrast to the filling box flow, ρ_e does not vary with time and advection of the density field must instead be balanced everywhere by diffusion of the stratifying species. Thus

$$W_e\frac{d\rho_e}{dz} = \frac{d}{dz}\left[\kappa^*\frac{d\rho_e}{dz}\right], \tag{8}$$

where κ^* is the diffusivity characterising the vertical transport of the stratifying species. We note at this point that a molecular diffusivity is not necessarily appropriate to the interior since turbulence may be supported internally within the system.

Boundary conditions

The plume is assumed to be forced purely by a localised (point) source of destabilising buoyancy at $z = 0$. The rate at which buoyancy is exchanged per unit box width with the plume at the surface is the buoyancy flux F_0. The volume flux across the boundary where the plume is forced is zero and, under a point source approximation, both

$$\left[RW_p\right]\big|_{z=0} = 0 \tag{9}$$

and

$$R(0) = 0. \tag{10}$$

Therefore,

$$W_e(0) = 0 \tag{11}$$

by equation (7). Equations (9) and (10) show that the vertical velocity in the plume W_p, and hence the entrainment velocity

U_e (by equation 4), must be zero at $z = 0$. We ensure that the flow in the container reaches a steady state by supplying stabilising buoyancy to a half-plume (against a vertical boundary) at a rate $F_0 = F_c$ per unit box width or to an isolated plume at a rate $2F_0 = F_c$ per unit box width, where F_c is given by equation (1). We assume the fluid to have a linear equation of state. As we have developed an interior model that neglects variations in the horizontal, we choose to distribute the stabilising flux uniformly over the forcing boundary at $z = 0$. Upon integrating the buoyancy flux per unit box width over the plume cross-section and taking the limit as $z \rightarrow 0$, the boundary condition may be written

$$F_0 = \frac{\sqrt{\pi}}{2\sqrt{2}} \left[RW_p g \left(\rho_p - \rho_e\right)\right]\Big|_{z=0} = gL\kappa^* \frac{d\rho_e(0)}{dz}. \qquad (12)$$

Equations (4) – (8) are strictly valid only for the part of the box volume that excludes the plume outflow. Here we assume that the outflow redistributes fluid from the plume over the horizontal boundary at $z = H$, whereupon the fluid becomes part of the interior. Thus the densities of fluid in the plume and in the interior must be equal at $z = H$, and we expect boundary conditions of zero buoyancy flux in each of the plume and the interior, i.e.

$$F(H) = \frac{\pi}{2} \left[RW_p g \left(\rho_p - \rho_e\right)\right]\Big|_{z=H} = gL\kappa^* \frac{d\rho_e(H)}{dz} = 0. \quad (13)$$

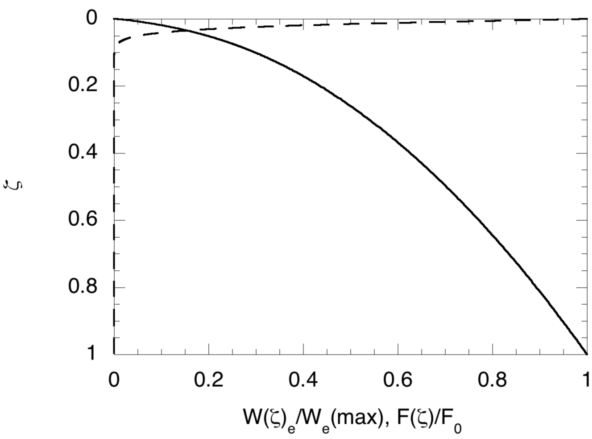

Figure 3: Profiles of vertical velocity (solid line) in the box interior and buoyancy flux (dashed line) in the plume predicted by the recycling box model with a localised cooling flux per unit box width of $q_c = 933$ W/m. Vertical velocities W_e have been normalised by the maximum vertical velocity $W_e(\mathrm{max}) = 0.12$ mm/s and the buoyancy flux has been normalised by the applied buoyancy flux $F_0 = 3.6 \times 10^{-4}$ N/m/s. The vertical coordinate is the normalised depth $\zeta = z/H$.

Solution

We solve numerically for R, W_p, ρ_p, W_e and ρ_e the system given by equations (4) – (8), and subject to the boundary conditions in equations (9) – (13). We note, however, that the plume outflow has been effectively neglected in the model here and the boundary condition in equation (13) cannot be enforced. In practice, we obtain a solution that satisfies the boundary conditions at $z = 0$ (equations 9 – 12). To a very good approximation, this solution is found to implicitly satisfy the boundary condition in equation (13) over the parameter range of interest. We verify this assumption below.

Comparison with laboratory data

The recycling box model solutions are presented by way of a comparison with available data from the laboratory experiment in figure 1 [5]. The experimental tank was 1.2 m long, 0.15 m wide and 0.2 m high, and a cooling flux per unit tank width $q_c = 933$ W/m was supplied over the right-hand half of the tank lid. By equation (1), this cooling flux generates a buoyancy flux per unit tank width of $F_c = F_0 = 3.6 \times 10^{-4}$ N/m/s, where we have taken $g = 9.81\,\mathrm{m}^2/\mathrm{s}$, the specific heat $c_p = 4184$ J/kg/K and the expansion coefficient $\alpha = 1.6 \times 10^{-4}\,{}^\circ\mathrm{C}^{-1}$.

We present in figures 3 and 4 the solutions for the buoyancy flux per unit width in the plume and the vertical velocity and temperature profile in the box interior, since these are the quantities of most practical interest. The solutions have been obtained using an entrainment coefficient $E = 0.1$ and a diffusivity that is approximately molecular $\kappa^* = 10^{-7}$ m^2/s. The vertical velocity in the box interior increases with depth owing to entrainment into the plume (equation 7). The buoyancy flux reduces very rapidly with depth owing to entrainment as the plume passes through the strong density stratification (the thermocline) at the top of the box. The thermocline thickness in this flow is predicted to be less than approximately one-tenth of the box depth, i.e. < 2 cm. Immediately below the thermocline the buoyancy flux is small and continues to decrease with depth (not visible on this scale). Thus it is apparent that the boundary condition at $z = H$ (equation 13) is indeed approximately satisfied by this solution. The variation with depth of the buoyancy flux is in stark contrast to the filling box flow, where F/F_0 reduces linearly from a value of one at the surface to zero at $z = H$.

As the vertical velocities in the tank are very small, only temperatures in the tank interior are readily measurable. Temperatures vary with horizontal position, particularly in the boundary layer adjacent to the applied heating and cooling fluxes. Therefore, to undertake a comparison with the temperature profile predicted by the recycling box model (which ignores horizontal variations in the box interior), we construct a horizontal average of a number of temperature profiles measured at several positions along the length of the tank. The averaged profile is also plotted in figure 4 for direct comparison with the theoretical prediction. Both the thermocline thickness and the top-to-bottom temperature difference are very well predicted. Although it is not visible in figure 4, both profiles show that the box interior (below the thermocline) supports a stable temperature stratification (with the exception in experiments of statically unstable regions immediately below the applied cooling). While the temperature gradient in the interior is many times smaller than in the thermocline, it is dynamically very significant since it forces the plume to penetrate to the bottom of the box.

The predicted maximum vertical velocities of 0.12 mm/s compare very well with numerical simulations undertaken by Mullarney *et al.* for the above experimental parameters [5, figure 10b]. On the basis of the predicted vertical velocity profile in figure 3, we would estimate the ventilation timescale $\tau \sim 2H/W_e(\mathrm{max}) \approx 30$ min to characterise the recycling of the tank volume through the plume. This timescale is consistent with observations of the time required for the dye in figure 1 to be spread throughout the tank.

Application to the oceans

Although the global oceans represent a very complex system that is not forced by buoyancy alone, models of horizontal convection can offer insight into the governing dynamics. Here we give only a brief outline of current work (to be presented elsewhere) examining the application of the recycling box model to the oceans. The global thermohaline circulation (or merid-

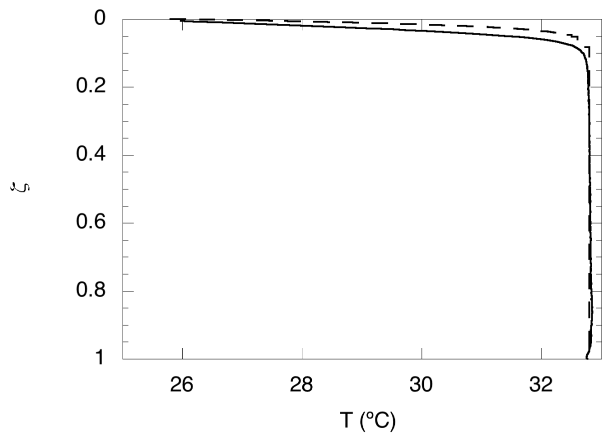

Figure 4: Comparison of temperature profiles predicted by the recycling box model (dashed line) for a line plume with data from the laboratory experiment (solid line) of Mullarney *et al.* [5]. The laboratory temperature profile represents a horizontal average of five vertical profiles taken at various horizontal positions along the length of the tank. The predicted temperature profile is referenced to the measured average temperature at $\zeta = 1$.

ional overturning circulation) of the oceans carries heated surface waters poleward, where they are cooled and sink in highly localised downwelling at polar latitudes. In the downwelling plume of largest buoyancy flux (the outflow from the Weddell Sea) the cold water sinks to the bottom, spreads through much of the oceans as Antarctic Bottom Water, and is eventually returned to the surface layers by a broad, slow upwelling in the interior. Laboratory experiments [4, 13] suggest that the plume reaching the bottom will dominate the overall density structure, hence we have developed a first order model using a single plume in the recycling box model. If the ocean is presumed to be in a steady state, the interior upwelling of cold water must be countered by downward mixing of heat (and upward mixing of density). The density structure has been similarly described by a simple one-dimensional balance of advection and diffusion (cf. equation 8), from which the measured abyssal gradients imply a diffusivity of order 10^{-4} m^{2}s^{-1} [6, 7]. The recycling box model may help to answer two important questions. Much of the heat transport is a consequence of wind-driven surface flows. However, it is not clear whether the heat transport is passive [7, 3, 14, 15], or whether instead the thermal buoyancy flux associated with the meridional heat flux is a significant contribution to the driving forces in the momentum budget for the mean overturning flow. The second question is whether the internal vertical mixing (which maintains the density structure of the oceans in the presence of the vertical components of the circulation) is substantially sustained by energy input from buoyancy-driven flow (hence by the heat flux) or whether the mixing is dominated by energy from tides and winds [6, 7, 3, 14, 15].

Conclusions

We have developed a model that describes ‘horizontal convection’ driven by a temperature gradient or heat flux applied along one horizontal boundary of a box. A steady circulation evolves, consisting of a localised turbulent plume that penetrates through the depth of the box and a broad gradual return flow to the plume. Important features in our steady model include zero net buoyancy input to the box and a balance between diffusion and advection of the density field in the box interior. The predictions from the model are in excellent agreement with available data from laboratory experiments and provide a sound basis from which to explore the relevance of the model to the global oceans.

References

[1] Baines, W. D. and Turner, J. S., Turbulent buoyant convection from a source in a confined region, *J. Fluid Mech.*, **37**, 1969, 51–80.

[2] Griffiths, R. W. and Hughes, G. O., The energetics of horizontal convection, *Proc. Fifteenth Aust. Fluid Mech. Conference*.

[3] Huang, R. X., Mixing and energetics of the oceanic thermohaline circulation, *J. Phys. Oceanogr.*, **29**, 1999, 727–746.

[4] Killworth, P. D. and Turner, J. S., Plumes with time-varying buoyancy in a confined region, *Geophys. Astrophys. Fluid Dyn.*, **20**, 1982, 265–291.

[5] Mullarney, J. C., Griffiths, R. W. and Hughes, G. O., Convection driven by differential heating at a horizontal boundary, *J. Fluid Mech.*, in press.

[6] Munk, W. H., Abyssal recipes, *Deep-Sea Res.*, **13**, 1966, 707–730.

[7] Munk, W. H. and Wunsch, C., Abyssal recipes II: energetics of tidal and wind mixing, *Deep-Sea Res. I*, **45**, 1998, 1977–2010.

[8] Pierce, D. W. and Rhines, P. B., Convective building of a pycnocline: laboratory experiments, *J. Phys. Oceanogr.*, **26**, 1996, 176–190.

[9] Rossby, H. T., On thermal convection driven by non-uniform heating from below: an experimental study, *Deep-Sea Res.*, **12**, 1965, 9–16.

[10] Rossby, H. T., Numerical experiments with a fluid non-uniformly heated from below, *Tellus*, **50**, 1998, 242–257.

[11] Stommel, H., On the smallness of sinking regions in the ocean, *Proc. Nat. Acad. Sci., Washington*, **48**, 1962, 766–772.

[12] Turner, J. S., *Buoyancy effects in fluids*, Cambridge University Press, Cambridge, England, 1973.

[13] Wong, A. B. D. and Griffiths, R. W., Stratification and convection produced by multiple plumes, *Dyn. Atmos. Oceans*, **30**, 2001, 101–123.

[14] Wunsch, C., Moon, tides & climate, *Nature*, **405**, 2000, 743–744.

[15] Wunsch, C. and Ferrari, R., Vertical mixing, energy, and the general circulation of the oceans, *Ann. Rev. Fluid Mech.*, **36**, 2004, 281–314.