Satellite Attitude Determination II

AERO4701 Space Engineering 3 – Week 7
Last Week

- Looked at the problem of attitude determination for satellites
- Examined several common methods such as inertial navigation, magnetometers, sun/star trackers, horizon scanners, attitude GPS etc.
- Looked at multi-sensor data fusion
Overview

• First Hour
  – Mathematical principles of attitude determination, Triad method, solving attitude using multiple line-of-sight vectors
Attitude Determination by Line-of-Sight Vectors

• Most common principle behind satellite attitude determination is by comparison of multiple vector measurements in body frame co-ordinates to known vectors in navigation frame co-ordinates

• We want to solve the Direction Cosine Matrix (DCM) satisfying:

\[ x_1^b = C_{a}^b x_1^a \]
\[ \vdots \]
\[ x_N^b = C_{a}^b x_N^a \]

• Where \( x_1^b, \ldots, x_N^b \) are the N measurements in body frame co-ordinates, \( x_1^a, \ldots, x_N^a \) are the N known vectors in navigation frame co-ordinates (generalised as ‘a’) and \( C_{a}^b \) is the DCM to solve
Attitude Determination by Line-of-Sight Vectors

- Problems breaks down to solving simultaneous vector transform equations
- DCM matrix has nine elements but can be parameterised using quaternions (4 parameters) or Euler angles (3 parameters)
- Each vector measurement gives two independent equations (unit vectors), thus one vector is insufficient and two or more vectors will result in an overdetermined system
Getting Vector Measurements: Magnetometers

- Three orthogonally aligned sensors for three-axis measurement: Can measure direction and magnitude of magnetic field, generally only direction used for attitude determination.
- Magnetic field reference model used for calculation of vector in ECI coordinates; then transformed into local navigation frame of interest.
- Magnetic field reference model can be anything from complicated (i.e. IGRF models) to simple dipole models.
Getting Vector Measurements: Sun Sensors

- Photocells (analog)
- Current output of photocell is proportionate to $\cos(\alpha)$
- Tri-axial arrangement commonly used for solving sun vector components w.r.t the sensor
- Sun direction computed in inertial co-ordinates based on time
Getting Vector Measurements: Star Trackers

- CCD element (digital)
- Position of star measured in pixels in image plane
- Unit vector direction computed using known camera calibration properties

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix} = \begin{bmatrix}
  f_u \left( \frac{y^s}{x^s} \right) + u_0 \\
  f_v \left( \frac{z^s}{x^s} \right) + v_0
\end{bmatrix}
\]

Pixels

\[f_u, f_v\] Focal length (in pixels) for each axis of the camera

AERO4701 Space Engineering 3 – Week 6
Solving Attitude from Two Vectors: Triad Method

- Simple deterministic way of computing direction cosine matrix using two vector measurements $s^b$ and $m^b$
- Construct reference frame $t$ components in body frame ($b$) and inertial frame ($i$) from vector measurements

- First basis vector along one of the measurement vectors

\[
\hat{t}_1 = \hat{s} \\
t_{1b} = s_b \\
t_{1i} = s_i
\]

- Second basis vector perpendicular to both measurements

\[
\hat{t}_2 = \hat{s} \times \hat{m} \\
t_{2b} = \frac{s_b \times m_b}{|s_b \times m_b|} \\
t_{2i} = \frac{s_i \times m_i}{|s_i \times m_i|}
\]

- Third basis vector perpendicular to other two basis vectors

\[
\hat{t}_3 = \hat{t}_1 \times \hat{t}_2 \\
t_{3b} = t_{1b} \times t_{2b} \\
t_{3i} = t_{1i} \times t_{2i}
\]
Solving Attitude from Two Vectors: Triad Method

- Construct transformation matrices by putting together basis vectors

\[ C_t^b = \begin{bmatrix} t_{1b} & t_{2b} & t_{3b} \end{bmatrix} \quad C_t^i = \begin{bmatrix} t_{1i} & t_{2i} & t_{3i} \end{bmatrix} \]

\[ C_i^b = C_t^b C_t^i = \begin{bmatrix} t_{1b} & t_{2b} & t_{3b} \end{bmatrix} \begin{bmatrix} t_{1i} & t_{2i} & t_{3i} \end{bmatrix}^T \]

- Can only use the information from two vectors using this method: assumes vector measurements must be very accurate

- Given that there are small errors in our measurements it is better to combine the information from more vectors
Using Multiple Vector Measurements: Newton’s Method

• In the same way Newton’s method was applied for solving GPS position from pseudoranges we can solve attitude using vector measurements.

• Solve for Euler angles:

\[
\begin{aligned}
\hat{m}^b_1 &= C_n^b \hat{m}^i_1 \\
\vdots \\
\hat{m}^b_N &= C_n^b \hat{m}^i_N
\end{aligned}
\]

\[
x = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}
\]

\[
\begin{bmatrix}
\hat{m}_{bx} \\
\hat{m}_{by} \\
\hat{m}_{bz}
\end{bmatrix} = 
\begin{bmatrix}
\cos \psi \cos \theta & \sin \psi \cos \theta & -\sin \theta \\
\cos \psi \sin \theta \sin \phi - \sin \psi \cos \phi & \sin \psi \sin \theta \sin \phi + \cos \psi \cos \phi & \cos \theta \sin \phi \\
\cos \psi \sin \theta \cos \phi + \sin \psi \sin \phi & \sin \psi \sin \theta \cos \phi - \cos \psi \sin \phi & \cos \theta \cos \phi
\end{bmatrix}
\begin{bmatrix}
\hat{m}_{ix} \\
\hat{m}_{iy} \\
\hat{m}_{iz}
\end{bmatrix}
\]
Using Multiple Vector Measurements: Newton’s Method

- Start off with initial estimate of Euler angles
  \[ x = x_0 \]
  \[ x_0 = \begin{bmatrix} \phi_0 & \theta_0 & \psi_0 \end{bmatrix} \]

- Solve for linearised error \( \Delta x \)
  \[ \hat{m}_{1:N}^b = f(x) \]
  \[ \hat{m}_{1:N_0}^b + \Delta \hat{m}_{1:N_0}^b = f(x_0 + \Delta x) \]

- Correct estimate

- Iterate until error converges to zero

\[ \Delta \hat{m}_{1:N}^b \approx \begin{bmatrix} \frac{\partial f}{\partial x} \end{bmatrix}_{x=x_0} \cdot \Delta x \]

\[ \Delta x = \left( H^T H \right)^{-1} H^T \Delta \hat{m}_{1:N}^b \]

\[ H^{-L} \]

\[ x = x + \Delta x \]
Newton’s Method: Jacobian Matrix

\[
\begin{align*}
H_{1,1} & = 0 \\
H_{1,2} & = -c_\psi s_\theta m_{ix} - s_\psi s_\theta m_{iy} - c_\theta m_{iz} \\
H_{1,3} & = -s_\psi c_\theta m_{ix} + c_\psi c_\theta m_{iy} \\
H_{2,1} & = (c_\psi s_\theta c_\phi + s_\psi s_\phi)m_{ix} + (s_\psi s_\theta c_\phi - c_\psi s_\phi)m_{iy} + c_\theta c_\phi m_{iz} \\
H_{2,2} & = c_\psi c_\theta s_\phi m_{ix} + s_\psi c_\theta s_\phi m_{iy} - s_\theta s_\phi m_{iz} \\
H_{2,3} & = (-s_\psi s_\theta s_\phi - c_\psi c_\phi)m_{ix} + (c_\psi s_\theta s_\phi - s_\psi c_\phi)m_{iy} \\
H_{3,1} & = (-c_\psi s_\theta s_\phi + s_\psi c_\phi)m_{ix} + (-s_\psi s_\theta s_\phi - c_\psi c_\phi)m_{iy} - c_\theta s_\phi m_{iz} \\
H_{3,2} & = c_\psi c_\theta c_\phi m_{ix} + s_\psi c_\theta c_\phi m_{iy} - s_\theta c_\phi m_{iz} \\
H_{3,3} & = (-s_\psi s_\theta c_\phi + c_\psi s_\phi)m_{ix} + (c_\psi s_\theta c_\phi + s_\psi s_\phi)m_{iy}
\end{align*}
\]
Multiple Vectors: Connections to Filtering

- Attitude estimation using multiple vectors in overdetermined system is analogous to curve fitting; we are finding the best fit of the Euler angles given all of the measurement samples.
- Sometimes all vector measurements might have varying degrees of accuracy (i.e. combine magnetometer measurement with a star tracked by a star tracker).
Multiple Vectors: Connections to Filtering

• We can weight the effect each vector measurement has on the iterated solution by using a weighted linear least squares solution in the iteration:

\[ \Delta x = \left[ (WH)^T (WH) \right]^{-1} (WH)^T W \Delta \hat{m}_b^{1:N} \]

• Where \( W \) is a diagonal matrix of weighting factors for each measurement
Example

- Three unit vector observations are made in body coordinates:
  \([-0.0102, 0.5852, -0.5894]^T, [-0.2680, 0.3595, 0.2428]^T, \]
  \([0.8555, -0.1442, 0.6224]^T\)

- We will add errors to the first measurement:
  \([0.01, -0.02, 0.04]^T\)

- The reference vectors are given in the LVLH frame:
  \([0.2, 0.7, -0.4]^T, [0.1, 0.3, 0.4]^T, [0.7, -0.8, 0.1]^T\)

- Use Newton’s method to solve for the Euler angles
Example

- True Euler angles are: 
  \[ [\phi, \theta, \psi] = [11^\circ, 32^\circ, -45^\circ] \]

- Angle estimates are: 
  \[ [\phi, \theta, \psi] = [10.668^\circ, 32.1723^\circ, -44.8664^\circ] \]

\[
\mathbf{W} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

- Angle estimates are: 
  \[ [\phi, \theta, \psi] = [10.9927^\circ, 32.0036^\circ, -44.9973^\circ] \]

\[
\mathbf{W} = \begin{bmatrix}
0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 \\
\end{bmatrix}
\]