Amme 3500: System Dynamics and Control
Frequency Domain Modelling

Dr. Stefan B. Williams

Course Outline

<table>
<thead>
<tr>
<th>Week</th>
<th>Date</th>
<th>Content</th>
<th>Assignment Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 Mar</td>
<td>Introduction</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>8 Mar</td>
<td>Frequency Domain Modelling</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>15 Mar</td>
<td>Transient Performance and the s-plane</td>
<td>Assign 1 Due</td>
</tr>
<tr>
<td>4</td>
<td>22 Mar</td>
<td>Block Diagrams</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>29 Mar</td>
<td>Feedback System Characteristics</td>
<td>Assign 2 Due</td>
</tr>
<tr>
<td>6</td>
<td>5 Apr</td>
<td>Root Locus</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>12 Apr</td>
<td>Root Locus 2</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>19 Apr</td>
<td>Bode Plots</td>
<td>No Tutorials</td>
</tr>
<tr>
<td>9</td>
<td>26 Apr</td>
<td>BREAK</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>3 May</td>
<td>Bode Plots 2</td>
<td>Assign 3 Due</td>
</tr>
<tr>
<td>11</td>
<td>10 May</td>
<td>State Space Modeling</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>17 May</td>
<td>State Space Design Techniques</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>24 May</td>
<td>Advanced Control Topics</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>31 May</td>
<td>Review</td>
<td>Assign 4 Due</td>
</tr>
</tbody>
</table>

A Familiar Mechanical Example

- In the last lecture, we considered this mechanical system
- We derived a differential equation defining this system
- How do we solve for $y(t)$?

$$\sum F = m\dddot{y}$$

$$m\dddot{y} = f(t) - Ky(t) - K_d\dot{y}(t)$$

$$m\dddot{y} + K_d\dot{y}(t) + Ky(t) = f(t)$$

How do we find $y(t)$?

- As we saw, the input-output relationship is usually expressed in terms of a differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1\frac{d y(t)}{dt} + a_0 y(t) = b_n\frac{d^n u(t)}{dt^n} + \cdots + b_1 u(t)$$

- If we can describe the characteristics of the plant using a general function, $h(t)$, we can compute the output $y(t)$ given some arbitrary input $u(t)$
LTI Systems

• In this course, we will consider Linear Time Invariant Systems
  – Linear: the output of the system is equal to the sum of the input responses
  – Time Invariant: the system’s dynamic characteristics are time independent

\[ y = \sum_{i=1}^{n} y_i \]

\[ y = \sum_{i=1}^{n} f(u_i) \]

The Unit Impulse

• Recall the impulse function, \( d(t) \)
• In the limit, we can represent an arbitrary function as a sum of impulses
• By the principle of superposition, if we can find the response of the system to an impulse, we will be able to find the response to an arbitrary input

\[ \int_{-\infty}^{\infty} f(\tau)d(\tau) d\tau = f(t) \]

Impulse Response

• The impulse response represents the system response to an impulse input
• We will examine techniques for calculating the impulse response shortly
  • We usually denote the impulse response of a system as \( h(t) \)

Convolusion Integral

• If we consider the input to a system as a sum of impulses, we can solve for the system output by integrating the signal with its impulse response over time
• This yields the system response to an arbitrary input signal \( u(t) \)

\[ y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau \]

\[ y(t) = u(t) * h(t) \]
Example: Solving in time

- Assume we have a system described by the following differential equation
  \[ \dot{y}(t) + ky(t) = u(t) \]
- The impulse response for this system is
  \[ h(t) = e^{-kt} \]
- We’ll see how we can find the impulse response shortly

Cascaded systems

- Now what happens if we have multiple components in the system?
- It clearly becomes difficult to manipulate these integrals beyond the simplest cases

The input signal as an Exponential

- If we further assume that the input signal can be represented by an exponential, \( e^{st} \)
  \[ y(t) = \int_{-\infty}^{\infty} u(t - \tau)h(\tau)d\tau \]
  \[ = \int_{-\infty}^{\infty} e^{s(t-\tau)}h(\tau)d\tau \]
  \[ = e^{st}\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau \]
- The constant \( s \) may be complex, expressed as \( s = \sigma + j\omega \)
The input signal as an Exponential

- This provides us with a rich basis function for describing functions.
- Through Euler’s formula, we find:
  \[ e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t} (\cos \omega t + j \sin \omega t) \]
- Fourier analysis tells us that this is sufficient for representing any signal.

The input signal as an Exponential

- Now let us replace the integral term by the constant \( H(s) \)

\[ y(t) = e^{st} \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau = H(s)e^{st} \]

where \( H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau \)

Laplace Transform

- In general

\[ F(s) = \int_{0}^{\infty} f(t)e^{-st} dt \]

Where \( s = \sigma + j\omega \)

\[ f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{st} ds \]

Table of Laplace Transforms

- What about that nasty integral in the Laplace operation?
- We normally use tables of Laplace transforms rather than solving the preceding equations directly.
- This greatly simplifies the transformation process:

<table>
<thead>
<tr>
<th>Item no.</th>
<th>( f(t) )</th>
<th>( F(s) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \delta(t) )</td>
<td>1</td>
</tr>
<tr>
<td>2.</td>
<td>( u(t) )</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>3.</td>
<td>( u(t) )</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>4.</td>
<td>( \sin(\omega t) )</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
</tr>
<tr>
<td>5.</td>
<td>( \cos(\omega t) )</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
</tr>
</tbody>
</table>
Laplace Transform Theorems

- The Laplace Transform is a linear transformation between functions in the \( t \) domain and \( s \) domain.

<table>
<thead>
<tr>
<th>No.</th>
<th>Theorems</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \mathcal{L}{f(t) - g(t)} = \left[ e^{-st} - e^{-st} \right] )</td>
<td>Definite</td>
</tr>
<tr>
<td>2</td>
<td>( \mathcal{L}{f(t)} = F(s) )</td>
<td>Linearity theorem</td>
</tr>
<tr>
<td>3</td>
<td>( \mathcal{L}{f(t) + g(t)} = F(s) + G(s) )</td>
<td>Linearity theorem</td>
</tr>
<tr>
<td>4</td>
<td>( \mathcal{L}{af(t)} = aF(s) )</td>
<td>Frequency shift theorem</td>
</tr>
<tr>
<td>5</td>
<td>( \mathcal{L}{f(t) - g(t)} = \left[ e^{-st} - e^{-st} \right] )</td>
<td>Time shift theorem</td>
</tr>
<tr>
<td>6</td>
<td>( \mathcal{L}{\frac{f(t)}{t}} = \frac{1}{s} )</td>
<td>Scaling theorem</td>
</tr>
<tr>
<td>7</td>
<td>( \mathcal{L}{\frac{f(t)}{t}} = \frac{1}{s} )</td>
<td>Differentiation theorem</td>
</tr>
<tr>
<td>8</td>
<td>( \mathcal{L}{af(t)} = aF(s) )</td>
<td>Differentiation theorem</td>
</tr>
<tr>
<td>9</td>
<td>( \mathcal{L}{af(t) + g(t)} = aF(s) + G(s) )</td>
<td>Differentiation theorem</td>
</tr>
<tr>
<td>10</td>
<td>( \mathcal{L}{\frac{f(t)}{t}} = \frac{1}{s} )</td>
<td>Integration theorem</td>
</tr>
<tr>
<td>11</td>
<td>( \mathcal{L}{\frac{f(t)}{t}} = \frac{1}{s} )</td>
<td>Initial value theorem</td>
</tr>
</tbody>
</table>

How does this help us?

- If the Laplace transform of the input can also be found, then the Laplace transform of the output is simply the product of these two fractions.
- We can then use partial fraction expansion to expand the response as a sum of simpler terms whose inverse LT can be found in the tables.

Comparing Solution Methods

- Starting with an impulse response, \( h(t) \), and an input, \( u(t) \), find \( y(t) \).
- Convolution:
  \[ u(t), \; h(t) \xrightarrow{ \text{convolution} } y(t) \]
- Multiplication, algebraic manipulation:
  \[ U(s), \; H(s) \xrightarrow{\text{multiplication} } Y(s) \]

\[ \mathcal{L}\{y(t)\} = \mathcal{L}\{u(t) * h(t)\} \]
\[ Y(s) = U(s)H(s) \]

\[ \mathcal{L}\{U(s)H(s)H_1(s)\} \]

\[ Y(s) = U(s)H(s)H_1(s) \]
Transfer Function

- The transfer function $H(s)$ of a system is defined as the ratio of the output of the system and input with zero initial conditions.
- This is effectively the Laplace transform of the impulse response for the system.

Example: Transfer Function

- Recall the system we examined earlier:
  \[ \dot{y}(t) + ky(t) = u(t) \]
- To find the transfer function for this system, we perform the following steps:
  \[ sH(s) + kH(s) = 1 \]
  \[ H(s) = \frac{1}{s + k} \quad \text{or} \quad h(t) = e^{-kt} \]

Forced vs. Natural Response

- The output response of a system is the sum of two responses:
  - The natural, or homogeneous, response describes the way the system dissipates or acquires energy. The form of this response is dependent on the system, not the input.
  - The forced, or particular, response represents the system response to a forcing function.
- Together these elements determine the overall response of the system.

A Familiar Mechanical Example

- Earlier, we considered this mechanical system.
- Now we can attempt to find $y(t)$.

\[ M\ddot{y}(t) + K\dot{y}(t) + Ky(t) = f(t) \]
\[ M\left( s^2Y(s) - sy(0) - \dot{y}(0) \right) + K\left( sY(s) - y(0) \right) + KY(s) = F(s) \]
Example: Natural Response

• The natural response of the system can be found by assuming no input force

\[
(Ms^2 + K_d s + K)Y(s) = (Ms + K_d)y(0) + M\dot{y}(0)
\]

\[
Y(s) = \frac{(Ms + K_d)y(0) + M\dot{y}(0)}{Ms^2 + K_d s + K}
\]

• Taking inverse Laplace transform will give us the system response

Example: Natural Response

• Suppose we wish to find the natural response of the system to an initial displacement of 0.5m

• Assume the mass of the system is 1kg and the systems constant \(k_d\) is 4 Nsec/m and \(k\) is 3 N/m

Example: Natural Response

• By Partial Fraction expansion we find

\[
Y(s) = \frac{0.5(s + 4)}{(s^2 + 4s + 3)}
\]

\[
= \frac{0.5(s + 4)}{(s + 1)(s + 3)}
\]

\[
= \frac{3/4}{s + 1} - \frac{1/4}{s + 3}
\]

\[
y(t) = 0.75e^{-t} - 0.25e^{-3t}
\]

Example: Natural Response

• What if we change the spring constant to 20 N/m?

\[
Y(s) = \frac{0.5(s + 4)}{(s^2 + 4s + 20)}
\]

\[
= \frac{0.5(s + 4)}{(s + 2)^2 + 16}
\]

\[
= 0.5\left(\frac{(s + 2)}{(s + 2)^2 + 16} + \frac{2}{(s + 2)^2 + 16}\right)
\]

\[
y(t) = 0.5e^{-2t}\left(\cos 4t + 0.5\sin 4t\right)
\]
Example : Forced Response

• Assuming zero initial conditions, the forced response will be

\[ (Ms^2 + K_d s + K)Y(s) = F(s) \]

\[ Y(s) = \frac{1}{Fs} \frac{1}{Ms^2 + K_d s + K} \]

• This is the transfer function for this system

Example : Forced Response

• Suppose we wish to find the response of the system to a unit step input force \( f(t) \)

• Assume the mass of the system is 1kg and the systems constant \( k_d \) is 4 Nsec/m and \( k \) is 3 N/m

\[ \frac{Y(s)}{F(s)} = \frac{1}{s^2 + 4s + 3} \]

\[ Y(s) = \frac{1}{s (s + 1)(s + 3)} \]

Example : Forced Response

• We have found

\[ Y(s) = \frac{1}{s(s + 1)(s + 3)} \]

\[ Y(s) = \frac{C_1}{s} + \frac{C_2}{s + 1} + \frac{C_3}{s + 3} \]

\[ C_1 = \frac{1}{(s + 1)(s + 3)} \bigg|_{s=0} = \frac{1}{2} \]

\[ C_2 = \frac{1}{s(s + 3)} \bigg|_{s=-1} = -\frac{1}{2} \]

\[ C_3 = \frac{1}{s(s + 1)} \bigg|_{s=-3} = \frac{1}{6} \]

• We wish to write \( Y(s) \) in terms of its partial-fraction expansion

Example : Forced Response

• So

\[ Y(s) = \frac{1}{3s} - \frac{1}{2(s+1)} + \frac{1}{6(s+3)} \]

• We now take inverse Laplace transforms of these simple transforms

\[ y(t) = -\frac{1}{2} e^{-t} + \frac{1}{6} e^{-3t} \]
Example : Forced Response

\[ y(t) = \frac{1}{3} - \frac{1}{2} e^{-t} + \frac{1}{6} e^{-3t} \]

- This represents the step response for this system
- This plot shows the response \( y(t) \) in time

Using Matlab to verify result

```matlab
% Example 1: Step response for
% Y(s)/F(s) = 1/(s^2 + 4s + 3)
% sys = tf(1,[1,4,3]);
% step(sys);
% hold on;
% pause;
%
% Solving for time domain response yields
% y(t) = \frac{1}{3} - \frac{1}{2}e^{-t} + \frac{1}{6}e^{-3t}
% t = 0:0.1:6;
% y_t = 1/3 - 1/2*exp(-t) + 1/6*exp(-3*t);
% plot(t, y_t, 'r');
% pause;
```

Poles, Zeros and System Response

- The poles of a transfer function are the values of the LT variables, \( s \), that cause the transfer function to become infinite
- The zeros of a transfer function are the values of the LT variables, \( s \), that cause the transfer function to become zero
- A qualitative understanding of the effect of poles and zeros on the system response can help us to quickly estimate performance

The s-plane

Recall

\[ Y(s) = \frac{1}{s(s+1)(s+3)} \]
\[ y(t) = \frac{1}{3} - \frac{1}{2} e^{-t} + \frac{1}{6} e^{-3t} \]
Example : Forced Response

• What if we change the spring constant $k$ to 20 N/m?

$$
Y(s) = \frac{1}{F(s)} = \frac{1}{s^2 + 4s + 20} = \frac{1}{s(s + 2 + 4j)(s + 2 - 4j)}
$$

$$
Y(s) = \frac{A}{s^2} + \frac{Bs + C}{(s + 2)^2 + 16}
$$

$$
1 = A(s^2 + 4s + 20) + Bs^2 + Cs
$$

$$
A = \frac{1}{20}, B = -\frac{1}{20}, C = -\frac{1}{5}
$$

The s-plane

• Now

$$
Y(s) = \frac{1}{s(s + 2 + 4j)(s + 2 - 4j)}
$$

$$
y(t) = \frac{1}{20} e^{-2t} (\cos 4t + 4 \sin 4t)
$$

Example : Forced Response

• Notice that the step response has changed in response to the change in the modified parameter $k$.

• The steady state response and transient behaviour are different

Using Matlab to verify result

```matlab
% Example 1: Step response for
% Y(s)/F(s) = 1/(s^2 + 4s + 20)
% sys = tf([1,1,4,20]);
% step(sys);
% hold on;
% pause;
```
Qualitative Effect of Poles

- A first order system without zeros can be described by:
  \[ H(s) = \frac{1}{s + \sigma} \]
- The resulting impulse response is:
  \[ h(t) = e^{-\sigma t} \]
- When
  - \( \sigma > 0 \), pole is located at \( s < 0 \), exponential decays = stable
  - \( \sigma < 0 \), pole is at \( s > 0 \), exponential grows = unstable

Step Response
  \[ y(t) = \frac{1}{\sigma} (1 - e^{-\sigma t}) \]

The s-plane

- From the preceding we can conclude that in general, if the poles of the system are on the left hand side of the s-plane, the system is stable.
- Poles in the right hand plane will introduce components that grow without bound.

Example: Automobile Suspension

- Consider the modelling of an automobile suspension system.
- Wheel can be modelled as a stiff spring with the suspension modelled as a spring and damper in parallel.
- Interested in describing the motion of the mass in response to changing conditions of the road.

Conclusions

- We have presented the motivation behind solving systems of differential equations.
- We have reviewed the Laplace Transform method for solving differential equations describing Linear Time Invariant systems.
- We have given a number of examples and have begun to investigate the effect of poles and zeros on the system response.
Further Reading

- Nise
  - Section 2.1-2.3 and 4.1-4.2
- Franklin & Powell
  - Section 3.1