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Robust Topology Optimization of Thin Plate Structure under Concentrated Load with Uncertain Load Point

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1. Abstract

This study investigates the robust topology optimization of the thin plate under concentrated load with uncertain load point. Several researches investigated the effect of uncertain load direction, load magnitude or load distribution on the topology optimization. However, the robust topology optimization considering uncertainty of the load point has not been studied yet. In this study, the load point uncertainty is modelled through the convex hull model. The nominal concentrated load in out-of-plane direction is applied at the center of the plate modeled based on Reissner-Mindlin plate theory. The load point uncertainty is limited in a circle centered at the nominal load point. The worst load condition is defined as the applied load at the worst point in the convex hull that gives the worst value of the mean compliance. The worst point is easily obtained from the convex hull approach. Then, the robust objective function is formulated as a weighted sum of the mean compliance obtained from the mean load condition and the worst compliance obtained from the worst load condition. This robust topology optimization is constructed using the level set-based topology optimization method. Through numerical examples, the robust optimum configuration is compared with the deterministic optimum configuration. Then, validity of the proposed robust design method is discussed.

2. Keywords: Level Set-Based Topology Optimization, Robust Optimization, Thin Plate Structure, Convex Hull, Worst Load Case

3. Introduction

Recently, the robust optimum design is widely applied to the field of engineering design problems that consider uncertainties of design parameters such as material constants and applied load conditions [1, 2]. Integrating the topology optimization and the robust design is generally called the robust topology optimum design. Several studies have been conducted on the robust topology optimization. Takezawa *et al.* [3] introduced the worst load condition of the applied load direction or the load distribution in the topology optimization. Chen *et al.* [4] applied the random field process to evaluate the space-varied random parameters. We proposed the robust topology optimization method [5] that integrates the level set-based topology optimization [6] and the sensitivity based robust optimization method [7]. Then, we applied the stationary stochastic process to model spatially-variable uncertain parameters for the robust topology optimization [8]. On the research, uncertain design parameters such as Young's modulus and distributed load with spatial distribution are modeled by using the stationary stochastic process with a reduced set of random variables.

This study considers the robust topology optimization for the thin plate structure. On the authors' previous study [9], deterministic level set-based topology optimization method for the thin-plate structure was proposed, where the bending plate is modeled based on Reissner-Mindlin theory, This study extends it to the robust topology optimization in consideration of the applied load point uncertainty. Under actual situation, the applied load point may be varied. Therefore, the variation of the applied load point is modeled by using the convex hull modeling [10]. The convex hull is applied to obtain the worst case of uncertain parameters. By approximating the uncertainty parameter range in the convex hull, the worst case is easily obtained. Then, the objective function is formulated as a weighted sum of the mean compliance by the mean applied load and the worst compliance that is given by the worst load condition. Through numerical example, the validity of the robust topology design is discussed.

4. Topology Optimization

4.1 Level Set-Based Topology Optimization

This study uses the level set-based topology optimization method [6]. The method can create holes in the solid domain during optimization by introducing energy term derived from the phase field theory. Additionally, the

method allows qualitative control of the geometry complexity of optimal configurations.
The level set function $\phi(\boldsymbol{x})$ is introduced to represent a clear shape boundary $\partial\Omega$ between the material domain Ω and the void domain $D\backslash\Omega$ as $\phi(\boldsymbol{x}) = 0$ where $\boldsymbol{x}$ indicates an arbitrary position in D. The level set function is defined to take a positive value in the material domain and negative in the void domain as follows:

$$\begin{cases} 0 < \phi(\boldsymbol{x}) \leq 1 & ^{\forall}\boldsymbol{x} \in \Omega\backslash\partial\Omega \\ \phi(\boldsymbol{x}) = 0 & ^{\forall}\boldsymbol{x} \in \partial\Omega \\ -1 \leq \phi(\boldsymbol{x}) < 0 & ^{\forall}\boldsymbol{x} \in D\backslash\Omega \end{cases} \tag{1}$$

The limit state function is bounded in $[-1,1]$ for introducing a fictitious interface energy based on the concepts of phase field method to the objective functional.
The design optimization is formulated as following equation that contains an objective functional $F(\Omega(\phi))$.

$$\inf_{\phi} \quad F(\Omega(\phi)) = \int_{\Omega} f(\boldsymbol{x}) \mathrm{d}\Omega \tag{2}$$

where $f(\boldsymbol{x})$ is the integrand function.
Since the above formulation allows to have discontinuous at every point, the regularization term is introduced based on the concept of phase field method [6].

$$\inf_{\phi} \quad F_R(\Omega(\phi)) = \int_{\Omega} f(\boldsymbol{x})\, \mathrm{d}\Omega + \int_{D} \frac{1}{2}\tau|\nabla\phi|^2 \mathrm{d}\Omega \tag{3}$$

$$\text{subject to} \quad G(\Omega) = \int_{\Omega} \mathrm{d}\Omega - V_{\max} \leq 0 \tag{4}$$

where F_R is a regularized objective functional, τ is a regularization parameter that represents the ratio of the fictitious interface energy, and $G(\Omega)$ indicates the volume constraint with the upper limit $V_{\max}$.
Using Eq (3) and (4), Lagrangian $\bar{F}_R$ is define as below:

$$\bar{F}_R(\Omega(\phi), \phi) = \int_{\Omega} f(\boldsymbol{x}) \mathrm{d}\Omega + \lambda G(\Omega(\phi)) + \int_{D} \frac{1}{2}\tau|\nabla\phi|^2 \mathrm{d}\Omega \tag{5}$$

The KKT conditions of the above optimization problem are derived as follows:

$$\bar{F}_R' = 0, \quad \lambda G = 0, \quad \lambda \geq 0, \quad G \leq 0 \tag{6}$$

where $\bar{F}_R$ and λ indicate the Lagrangian and the Lagrange multiplier, respectively.

4.2 Updating the Level Set Function

Level set function that satisfies the KKT conditions in Eq. (6) is candidate solutions of the optimization problem. Introducing a fictitious time t, and assuming that the variation of the level set function with respect to the time t is proportional to the gradient of Lagrangian, as follows:

$$\frac{\partial\phi}{\partial t} = -K(\phi)\bar{F}_R' \tag{7}$$

where $K(\phi) > 0$ is the positive proportionality coefficient.
Substitute Eq. (5) into Eq. (7) and applying Dirichlet boundary condition to the body domain boundary ∂D_N and Neumann boundary condition to the other boundary, the following time evolution equation is obtained:

$$\begin{aligned} \frac{\partial\phi}{\partial t} &= -K(\phi)\left(\bar{F}' - \tau\nabla^2\phi\right) \\ \frac{\partial\phi}{\partial n} &= 0 \qquad\qquad \text{on } \partial D\backslash\partial D_N \\ \phi &= 1 \qquad\qquad \text{on } \partial D_N \end{aligned} \tag{8}$$

where $H(\phi)$ is Heaviside function. Note that Eq. (8) is a reaction-diffusion equation, and the smoothness of the level set function is ensured. Further details are provided in [6].

5. Robust Topology Optimization

Robust optimum design considers the effect of uncertainty of design variables and parameters on the objective function and constraints. As shown in Fig. 1, the robust optimum design has smaller deterioration of the performance under variation of design parameters than that of the deterministic optimum design where z_0 and Δz denote nominal value and variation of design parameter, respectively.

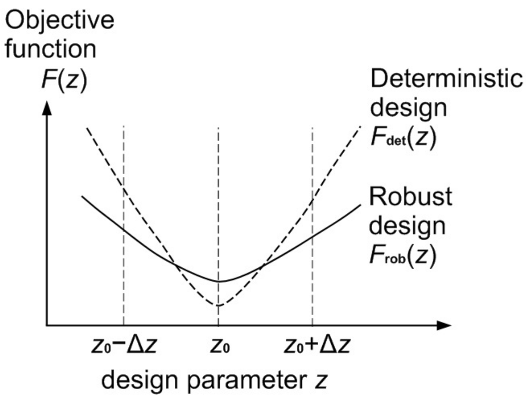


Figure 1: Concept of robust optimization

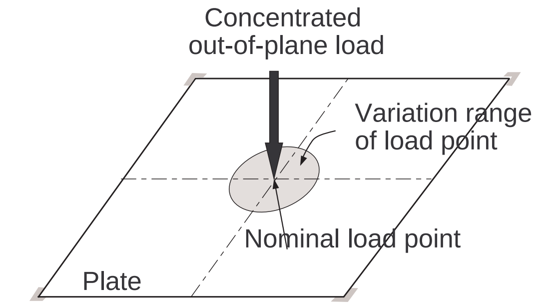


Figure 2: Variation range of applied load point on the plate

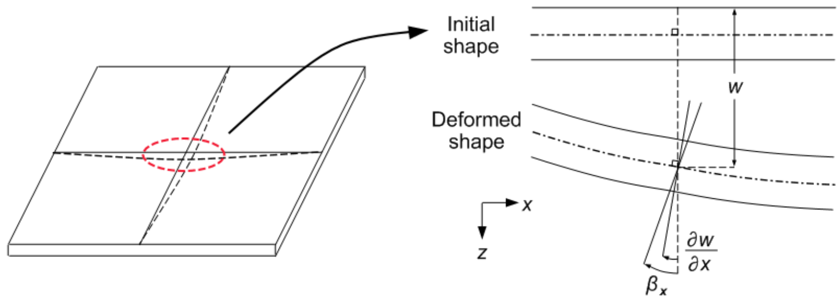


Figure 3: Reissner-Mindlin assumption

5.1 Design Problem of Thin Plate Structure

This study considers the variation of the load point of the applied concentrated load. As shown in Fig. 2, the square plate with fixed four vertices with applied the concentrated out-of-plane load is considered. The load point is modeled as uncertain parameter, where the nominal point is set at the center and the variation range is limited inside of the circle.

The conventional topology design problem is to minimize the mean compliance. That is, by using the strain energy $a(\boldsymbol{u},\boldsymbol{v})$ and the mean compliance $l(\boldsymbol{u})$, the objective functional is defined as follows:

$$\inf_{\Omega} \; : F(\Omega) = l(\boldsymbol{u}) \tag{9}$$

$$\text{subject to } \; : a(\boldsymbol{u},\boldsymbol{v}) = l(\boldsymbol{v}) \quad \text{for } {}^{\forall}\boldsymbol{v},\ \boldsymbol{u} \in U \tag{10}$$

where $a(\boldsymbol{u},\boldsymbol{v})$ and $l(\boldsymbol{v})$ are defined as follows:

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_{\Omega} \varepsilon(\boldsymbol{u}) : \boldsymbol{E} : \varepsilon(\boldsymbol{v}) d\Omega \tag{11}$$

$$l(\boldsymbol{v}) = \int_{\Gamma_t} \boldsymbol{t} \cdot \boldsymbol{v} d\Gamma \tag{12}$$

where ε is the linearized strain tensor, $\boldsymbol{E}$ is the elasticity tensor and the U is defined as follows:

$$U = \left\{ \boldsymbol{v} = v_i e_i \; : \; v_i \in H^1(D) \right\} \text{ with } \boldsymbol{v} = 0 \text{ in } \Gamma_u \tag{13}$$

Based on Reissner-Mindlin theory, the strain energy for the thin plate structure is described as follows:

$$\frac{1}{2} a(\boldsymbol{u},\boldsymbol{u}) = \frac{1}{2} \iint \left\{ M_x \frac{\partial \beta_x}{\partial x} + M_y \frac{\partial \beta_y}{\partial y} + M_{xy} \left(\frac{\partial \beta_y}{\partial x} + \frac{\partial \beta_x}{\partial y} \right) + Q_x \left(\frac{\partial w}{\partial x} + \beta_x \right) + Q_y \left(\frac{\partial w}{\partial y} + \beta_y \right) \right\} dxdy \tag{14}$$

where M_x, M_y, M_{xy} are the bending and the torsional moments, Q_x and Q_y are the shear force, β_x and β_y are the rotational angle, and h is the plate thickness as shown in Fig. 3.

The worst load case is defined as the load case that gives the worst value of the mean compliance in the given convex hull. In this study, the worst case is found to lie on the boundary on the convex hull by preliminary analysis. Therefore, the convex hull model is adopted. The worst case is easily obtained as solving the sub-optimization problem in each iteration of the topology optimization loop.

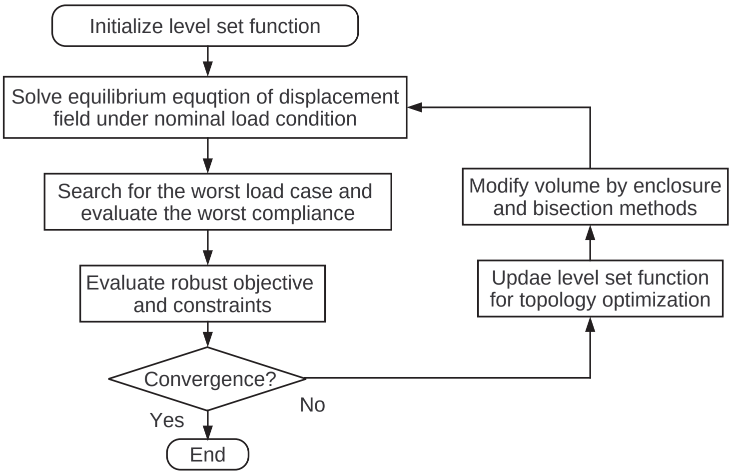


Figure 4: Flowchart of robust topology optimization

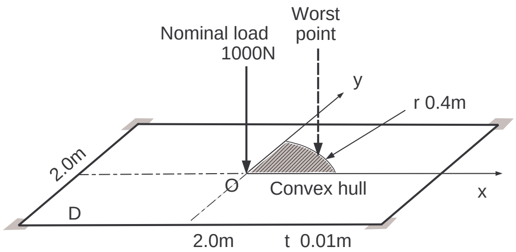


Figure 5: Plate model and convex hull for uncertain load point

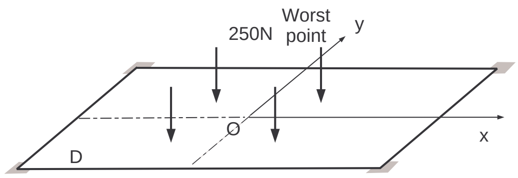


Figure 6: Plate model for robustness evaluation

5.2 Robust Topology Optimization

The objective of the robust topology optimization is defined as a weighted sum of the mean and the worst compliance as follows:

$$f_{\text{robust}}(\boldsymbol{x}) = (1-\alpha)a_{\text{nom}}(\boldsymbol{u},\boldsymbol{u}) + \alpha a_{\text{worst}}(\boldsymbol{u},\boldsymbol{u}) \tag{15}$$

where $a_{\text{nom}}(\boldsymbol{u},\boldsymbol{u})/2$ is the strain energy density under the deterministic nominal load, and $a_{\text{worst}}(\boldsymbol{u},\boldsymbol{u})/2$ is the strain energy density under the worst load in the convex hull, and $0 < \alpha < 1$ is a positive weighting coefficient.

The computational flow of the proposed robust topology optimization method is shown in Fig. 4. Starting the initialization of the level set function, the equilibrium equation of the nominal load is solved using FEM to evaluate the mean compliance. Then, the worst load condition is searched in the convex hull and the robust objective function is evaluated. After the convergence cheek, the level set function is updated. Then, the volume is modified to fit the upper limit by the enclosure and bisection method [8]. The equilibrium equation is solved for the updated geometry and the process is repeated until convergence.

6. Numerical Examples

As a simple numerical example, the square plate with 2.0m on a side and 0.01m in thickness with the fixed four vertices as shown in Fig. 5 is considered as a fixed design domain D. Young's modulus and Poisson's ratio are set as 210GPa, and 0.33, respectively. The fixed design domain D is discretized to 21850 elements for evaluating the mean and the worst compliance. For the topology optimization, the regularization parameter τ and the volume constraint are set as 5.0×10^{-5} and 50%, respectively.

The concentrated out-of-plane load of 1000N is applied at the center of the plate as a nominal load point. As a random parameter, the load point is assumed to be varied in the circle centered at the nominal load point. The convex hull is set as the quarter sector shown in Fig. 5, because of considering the symmetry condition,

It is expected that the unsymmetric load condition for the worst case will yield the unsymmetric optimum configuration. However, since the uncertain point will be lie on the other sectors, the unsymmetric configuration is not suitable as the robust optimum configuration. Therefore, the other three symmetric points to the worst load point are also considered as the worst load points. For evaluating the worst compliance a_{worst} in Eq. (15), the plate model is arranged to apply the four concentrated loads of four divided magnitude of 250N at the worst points as shown in Fig. 6.

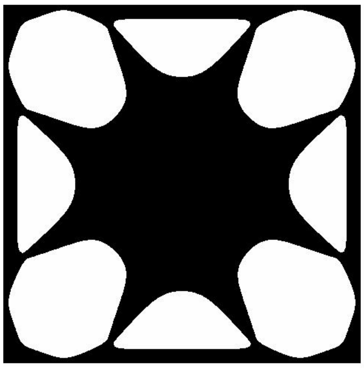


Figure 7: Deterministic optimum configuration

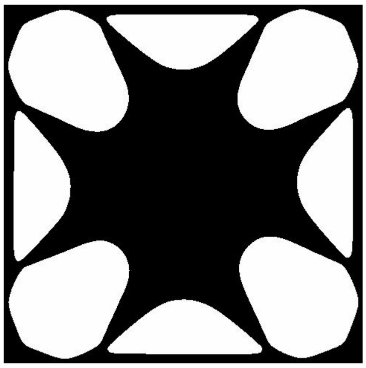


Figure 8: Robust optimum configuration ($\alpha = 0.70$)

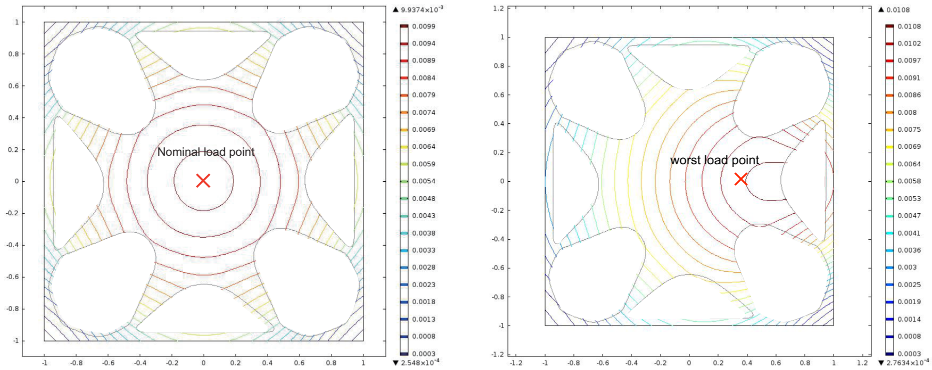
Table 1: Mean compliance under deterministic load

Configuration	Load point [m]	Mean Compliance [J/m^3]
Deterministic	(0.0, 0.0)	7.606×10^4
Robust ($\alpha = 0.30$)	(0.0, 0.0)	7.572×10^4
Robust ($\alpha = 0.50$)	(0.0, 0.0)	7.559×10^4
Robust ($\alpha = 0.70$)	(0.0, 0.0)	7.545×10^4

Table 2: Mean compliance under worst load

Configuration	Load point [m]	Mean Compliance [J/m^3]	Increase rate [%]
Deterministic	(0.0, 0.40)	8.127×10^4	6.848
Robust ($\alpha = 0.30$)	(0.0, 0.40)	8.048×10^4	6.292
Robust ($\alpha = 0.50$)	(0.0, 0.40)	8.016×10^4	6.049
Robust ($\alpha = 0.70$)	(0.40, 0.0)	7.984×10^4	5.813
Deterministic	(0.40, 0.0)	8.126×10^4	6.832
Robust ($\alpha = 0.30$)	(0.40, 0.0)	8.047×10^4	6.284
Robust ($\alpha = 0.50$)	(0.40, 0.0)	8.008×10^4	5.951
Robust ($\alpha = 0.70$)	(0.0, 0.40)	7.974×10^4	5.680

The deterministic optimum configuration obtained under the nominal load condition is shown in Fig. 7. The robust optimum configuration under $\alpha = 0.7$ in Eq. (15) is shown in Fig. 8. These configurations are very similar with each other except for the hole shape closed to the vertices. Fig. 8 shows the robust design under the case of $\alpha = 0.7$. The other optimum configurations for the smaller values of the weighting factors are almost the same in Fig. 8.
Table 1 compares the mean compliance values under the deterministic load between the deterministic and the robust optimum configurations with $\alpha = 0.3, 0.5$ and 0.7. It is found that the mean compliance of the robust configuration under the deterministic load is smaller than that of the deterministic configuration.
Then, Table 2 compares the mean compliance values under the worst load conditions. The load point shows the worst load point. The values of the mean compliance under the rotationally symmetric load point are also listed. The deterioration of the compliance value are almost the same between the deterministic and the robust configurations, though the deterioration rates of the robust configurations are smaller than that of the deterministic configuration. It means that the deterministic optimum configuration has higher robustness in this case. That's why the optimum configurations are the similar configurations.
It is expected that the research concerning the robust optimum design will expect to obtain the different design from the deterministic one. However, that is not always true. We must consider the effect of the random parameter on the deterministic optimum configuration first.
For the purpose, the out-of-plane deformation distributions for the deterministic optimum configuration are compared between the nominal and the worst load conditions in Fig. 9. The maximum displacement occurs at the load point under the nominal case. On the other hand, the maximum occurs not at the load point under the worst load case, at the righter position from the worst load point, at the edge of the hole. It is considered that the hole makes the maximum displacement position shift form the worst load point to the edge of the hole, that will make the deterioration of the worst compliance smaller. As a result, the deterministic optimum configuration happens to be



(a) Nominal load (b) Worst load

Figure 9: Out-of-plane deformation distribution for deterministic optimum configuration

robust for variations of the load point in this example.

7. Conclusion

This paper investigates the robust topology optimum design for the thin plate structure under the concentrated load with uncertain load point. The uncertainty is modeled by using the convex hull to find the worst load condition that yield the worst value of the mean compliance. The robust objective function is formulated as a weighted sum of the mean and the worst compliance. The optimum configuration is obtained by using the level set-based topology optimization.

Through the numerical examples, the robust configuration is almost similar to the deterministic configuration. It means that we must consider the effect of uncertainties of the design parameters on the deterministic optimum configuration at first.

We will investigate the effect of the other design parameters on the optimum configuration for the thin plate structure.

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