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Enhanced second-order reliability method and stochastic sensitivity analysis using importance sampling

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1. Abstract
The enhanced second-order reliability method (eSORM) is proposed in this study in order to improve accuracy in estimating a probability of failure. Conventional SORM additionally approximates an already approximated quadratic performance function to a parabolic surface, indicating that those methods are based on an incomplete second-order Taylor expansion of the performance function. This additional approximation means a loss of accuracy in estimating the probability of failure. The proposed SORM utilizes the importance sampling to calculate the probability of failure of a complete second-order Taylor expansion of the performance function without the parabolic approximation, so it shows better accuracy compared to the conventional SORM methods. The proposed SORM method also utilizes an approximated Hessian of the performance function by using the symmetric rank-one update in Quasi-Newton method, which means that additional function calls are not required except the computation used for MPP search. In addition to the improvement of the accuracy, stochastic sensitivity analysis is performed in the proposed method by applying the importance sampling to the quadratically approximated performance function. Therefore, the second-order sensitivity of the probability of failure as well as the first-order one can be easily computed in the proposed method without additional function calls.
2. Keywords: Reliability analysis, Second-order reliability method, Stochastic sensitivity analysis, Importance sampling, Approximated Hessian

3. Introduction
In a reliability analysis, it is quite difficult to estimate the probability of failure defined as a multi-dimensional integration over a nonlinear domain in a real engineering problem especially including finite element analysis. Hence, reliability methods based on function approximation are commonly used such as first-order reliability method (FORM) [1], second-order reliability method (SORM) [2-5], and dimension reduction method (DRM) [6-8]. Those methods approximate the performance function at the most probable point (MPP) which has the highest probability density on a limit-state surface and can be obtained by searching the minimum distance from the origin to the limit-state surface in the standard normal space (U-space). FORM which linearizes the performance function at MPP is the most commonly used reliability method due to its numerical efficiency. FORM shows reasonable accuracy when the performance function is almost linear or mildly nonlinear. However, FORM might give erroneous reliability estimation if the performance function is highly nonlinear. More accurate reliability estimation can be performed using SORM even for a highly nonlinear system since curvature of the performance function near MPP is considered in SORM by calculating second-order derivatives of the performance function. In spite of the fact that SORM is obviously more accurate than FORM, SORM is limitedly used in engineering problems due to the calculation of the second-order derivatives of the performance function, which might require huge computational cost.
After the second-order Taylor series of the performance function is constructed at MPP using the first and second-order derivatives of the performance function, the approximated function is once more approximated to a parabolic surface, indicating that the incomplete Taylor series is used in conventional SORM methods [9]. Furthermore, to obtain analytical formulation to calculate the probability of failure, an additional approximation such as an asymptotic approximation is introduced in the conventional SORM method. These two approximations mean a loss of accuracy in estimating the probability of failure.
To calculate the probability of failure more accurately without approximations additional to the quadratic approximation, the enhanced SORM (eSORM) method is proposed in this study. The proposed eSORM utilizes the importance sampling to calculate the probability of failure of the complete second-order Taylor expansion of the performance function without the parabolic approximation. Thus, it shows better accuracy compared to the conventional SORM methods. Sampling methods [10-12] such as the Monte Carlo Simulation (MCS) and the importance sampling estimate readily the probability of failure using the stochastic sampling because the complex analytical formulation is not required in the sampling method. In addition to the calculation of the probability of failure, stochastic sensitivity analysis is also readily performed without additional function calls in the sampling method [13]. However, the computational demand for the sampling method is generally prohibitive if an

engineering problem including virtual simulation such as finite element analysis is considered. Since the proposed eSORM applies the importance sampling not to the original performance function but to quadratically approximated function, the required computational cost except for the Hessian calculation is negligible. The proposed eSORM method also utilizes an approximated Hessian of the performance function by using the symmetric rank-one update in Quasi-Newton method, which means that additional function calls are not required except the computation used for MPP search [14]. In addition to the improvement of the accuracy, the stochastic sensitivity analysis is performed in the proposed method by applying the stochastic sensitivity analysis of the sampling method to the quadratically approximated performance function. The second-order sensitivity of the probability of failure as well as the first-order one can be easily computed in the proposed method without additional functional calls.

4. Enhanced second-order reliability method (eSORM)

4.1. FORM and SORM

The probability of failure can be defined as a multi-dimensional integral as [15]

$$P_f = P[G(\mathbf{X}) > 0] = \int_{G(\mathbf{X})>0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{1}$$

where $P[\cdot]$ is a probability function, X_i is a random variable and x_i is a realization of X_i. $f_{\mathbf{X}}(\mathbf{x})$ is a joint probability density function of $\mathbf{X}$, and $G(\mathbf{X})$ is a performance function such that $G(\mathbf{X}) > 0$ is defined as failure and $G(\mathbf{X}) = 0$ is defined as a limit-state equation. Due to difficulties in computing the multi-dimensional integral in Eq. (1), FORM linearizes the performance function $G(\mathbf{X})$ at a most probable point (MPP) $\mathbf{u}^*$ in U-space obtained by the transformation [16] as

$$G(\mathbf{X}) = g(\mathbf{U}) \cong g(\mathbf{u}^*) + \nabla g^T (\mathbf{U} - \mathbf{u}^*) \tag{2}$$

FORM is the most commonly used reliability analysis method due to the computational efficiency, and it shows reasonable accuracy in calculating the probability of failure for linear and mildly nonlinear performance functions. However, the error incurred by the linearization becomes considerable when the performance function is highly nonlinear. For this reason, SORM approximates the performance function quadratically given as [9]

$$g_Q(\mathbf{U}) = g(\mathbf{u}^*) + \nabla g^T (\mathbf{U} - \mathbf{u}^*) + \frac{1}{2}(\mathbf{U} - \mathbf{u}^*)^T \mathbf{H}(\mathbf{U} - \mathbf{u}^*) \tag{3}$$

where $\mathbf{H}$ is the Hessian matrix evaluated at MPP in U-space. Equation (3) is transformed to V-space and is rewritten as

$$\frac{\tilde{g}_Q(\mathbf{V})}{\|\nabla g\|} = V_N - \beta + \tilde{\mathbf{V}}^T \tilde{\mathbf{A}} \tilde{\mathbf{V}} + \frac{\beta^2}{2} \boldsymbol{\alpha} \frac{\mathbf{H}}{\|\nabla g\|} \boldsymbol{\alpha} - \beta \boldsymbol{\alpha}^T \frac{\mathbf{H}}{\|\nabla g\|} \mathbf{R}\mathbf{V} + 2V_N \mathbf{A}_{N1} \tilde{\mathbf{V}} + A_{NN} V_N^2 \tag{4}$$

where $\tilde{\mathbf{V}} = \{V_1, V_2, ..., V_{N-1}\}^T$, $\mathbf{A} = \frac{\mathbf{R}^T \mathbf{H} \mathbf{R}}{2\|\nabla g\|} = \begin{bmatrix} \tilde{\mathbf{A}} & \mathbf{A}_{1N} \\ \mathbf{A}_{N1} & A_{NN} \end{bmatrix}$ and V-space is transformed using the orthogonal transformation $\mathbf{u} = \mathbf{R}\mathbf{v}$. $\mathbf{R}$ is an $N \times N$ orthogonal rotation matrix whose N^{th} column is $\boldsymbol{\alpha} = \frac{\nabla g}{\|\nabla g\|}$, and computed using Gram-Schmidt orthogonalization such that $\mathbf{R}$ is written as $\mathbf{R} = [\mathbf{R}_1 \mid \boldsymbol{\alpha}]$ where $\boldsymbol{\alpha}^T \mathbf{R}_1 = \mathbf{0}$. In order to obtain the closed-form formula of the probability of failure, second-order approximated function in Eq. (4) is further approximated to a parabolic surface by neglecting all cross terms between $\tilde{\mathbf{V}}$ and V_N as follows

$$\frac{\tilde{g}_Q(\mathbf{V})}{\|\nabla g\|} = V_N - \beta + \tilde{\mathbf{V}}^T \tilde{\mathbf{A}} \tilde{\mathbf{V}} \tag{5}$$

Based on the parabolic approximation in Eq. (5), Breitung [3] proposed a simple formula for the probability of failure with additional approximations such as an asymptotic approximation, which is given by

$$P_f^{SORM} \cong \Phi(-\beta)\left\|\mathbf{I}_{N-1} - 2\beta\tilde{\mathbf{A}}\right\|^{-\frac{1}{2}} \tag{6}$$

In SORM, errors due to the approximations can be categorized as follows [9]

Type 1: Error due to approximating the performance function by the second-order Taylor expansion.

Type 2: Error due to further approximating the second-order approximated function to the parabolic surface.

Type 3: Error due to the additional approximations to obtain the closed-form formula for the probability of failure.

The error of type 1 cannot be eliminated unless higher-order Taylor expansion is constructed. The parabolic approximation in the error of type 2 means that the conventional SORM methods have been based on an incomplete second-order Taylor expansion, and the approximation in type 3 leads to the additional loss of accuracy in estimating the probability of failure.

4.2 The proposed eSORM

In order to improve accuracy in estimating probability of failure in SORM, the enhanced second-order reliability method (eSORM) is proposed in this study by utilizing the stochastic sampling method. Since eSORM calculates the probability of failure based on the complete second-order Taylor expansion of a performance function, its accuracy is considerably improved compared to the conventional SORM methods. The approximated Hessian of the performance function is used in this study, indicating that additional function calls are not required except the computation used for MPP search [14]. Furthermore, the second-order sensitivity of the probability of failure as well as the first-order sensitivity can be readily computed in the proposed eSORM.

By introducing an indicator function for failure region, the probability of failure in Eq. (1) can be rewritten as its expectation as [17]

$$P_f = \int_{\mathbf{R}^{\text{nrv}}} I_{\Omega_f}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x} = E\left[I_{\Omega_f}(\mathbf{X})\right] \tag{7}$$

where nrv means the number of random variables and $I_{\Omega_f}(\mathbf{x})$ is the indicator function which is defined as

$$I_{\Omega_f}(\mathbf{x}) \equiv \begin{cases} 1, & \mathbf{x} \in \Omega_f \\ 0, & otherwise \end{cases} \tag{8}$$

where Ω_f is the failure region which is defined as $G(\mathbf{X}) > 0$. In order to calculate the probability of failure in Eq. (7) using the stochastic sampling method, numerous function calls are required which is computationally very expensive and almost impossible if the computer simulation is included. Hence, instead of the original performance function, the quadratically approximated performance function defines the failure region in Eq. (8) in the proposed method as follows

$$I_{\hat{\Omega}_f}(\mathbf{x}) \equiv \begin{cases} 1, & \mathbf{x} \in \hat{\Omega}_f \\ 0, & otherwise \end{cases} \tag{9}$$

In Eq. (9), the failure region $\hat{\Omega}_f$ is defined using the second-order Taylor expansion of the performance function as

$$\hat{\Omega}_f \equiv \left\{\mathbf{x} : G_Q(\mathbf{x}) > 0\right\} \tag{10}$$

where $G_Q(\mathbf{x})$ is the quadratic performance function approximated in the original space (X-space) at MPP. Since a reference point of the second-order Taylor expansion is MPP in SORM, the approximation is accurate near MPP and is the most accurate at MPP. Hence, the importance sampling method selecting MPP as a sampling center is performed to calculate the probability of failure about the quadratic performance function $G_Q(\mathbf{x})$ in this study as

$$P_f = \int_{\mathbf{R}^{nrv}} I_{\hat{\Omega}_f}(\mathbf{x}) \frac{f_{\mathbf{X}}(\mathbf{x})}{h(\mathbf{x})} h(\mathbf{x})\, d\mathbf{x} = E\left[I_{\hat{\Omega}_f}(\mathbf{X}) \frac{f_{\mathbf{X}}(\mathbf{x})}{h(\mathbf{x})} \right] \tag{11}$$

where $h(\mathbf{x})$ is an instrumental density function for the importance sampling.

The proposed eSORM is based on the complete second-order Taylor expansion without the parabolic approximation, indicating that the type 2 error can be readily eliminated. The importance sampling is performed using the approximated performance function, not the original one in eSORM. Therefore, the computational cost to calculate the probability of failure after constructing the approximated function is negligible even though sufficient number of samples are evaluated. Considering that accurate calculation of probability of failure is possible in the sampling method with sufficient samples, the type 3 error incurred during obtaining the closed-form formula for the probability of failure in the conventional SORM is also easily removed. In this way, the proposed eSORM contains only the type 1 error and its accuracy is thus significantly improved even though the total computational cost of eSORM is the same as one of the conventional SORM.

Even though SORM usually shows great accuracy in estimating probability of failure, the calculation of the Hessian of a performance function aggravates the numerical burden of SORM. When SORM is used within RBDO, this problem intensifies due to the repeated reliability assessment. Hence, the proposed eSORM utilize approximated SORM [14] recently proposed in order to resolve the heavy computational burden in SORM by using the quasi-Newton approach to approximate the Hessian. Since the approximated Hessian is used instead of calculating the true Hessian, the additional function evaluation except for the MPP search is not required in order to construct the second-order Taylor expansion of the performance function.

4.3. The proposed stochastic sensitivity analysis

This study presents the sensitivity analysis of eSORM, so that eSORM is readily utilized to obtain the accurate reliability-based optimum in RBDO. Since the proposed eSORM is based on the stochastic sampling method, the stochastic sensitivity analysis using eSORM can be performed efficiently and simply using the stochastic sensitivity analysis of the stochastic sampling method. In the stochastic sampling method, the sensitivity of the probability of failure with respect to the mean of random variables is given by [17]

$$\frac{\partial P_f}{\partial \mu_j} = \int_{\mathbf{R}^{nrv}} I_{\Omega_f}(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_j} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} = E\left[I_{\Omega_f}(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_j} \right] \tag{12}$$

where $\frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_j}$ is known as the first-order score function which can be analytically obtained. If the random variables are correlated, the score function can be evaluated using a copula function. In a similar fashion with Eq. (11), Eq. (12) can be re-written using the instrumental density function and the failure region defined using the second-order Taylor expansion of the performance function in Eq. (10) as

$$\frac{\partial P_f}{\partial \mu_j} = \int_{\mathbf{R}^{nrv}} I_{\hat{\Omega}_f}(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_j} \frac{f_{\mathbf{X}}(\mathbf{x})}{h(\mathbf{x})} h(\mathbf{x}) d\mathbf{x} = E\left[I_{\hat{\Omega}_f}(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_j} \frac{f_{\mathbf{X}}(\mathbf{x})}{h(\mathbf{x})} \right] \tag{13}$$

In Eq. (13), the instrumental density function $h(\mathbf{x})$ is independent with the design variable μ_j.

The second-order sensitivity of the probability of failure in eSORM can be readily calculated in a similar way as

$$\begin{aligned} \frac{\partial^2 P_f}{\partial \mu_i \partial \mu_j} &= \int_{\mathbf{R}^{nrv}} I_{\hat{\Omega}_f}(\mathbf{x}) \left[\frac{\partial^2 \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_i \partial \mu_j} + \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_j} \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_i} \right] \frac{f_{\mathbf{X}}(\mathbf{x})}{h(\mathbf{x})} h(\mathbf{x}) d\mathbf{x} \\ &= E\left[I_{\hat{\Omega}_f}(\mathbf{x}) \left[\frac{\partial^2 \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_i \partial \mu_j} + \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_j} \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_i} \right] \frac{f_{\mathbf{X}}(\mathbf{x})}{h(\mathbf{x})} \right] \end{aligned} \tag{14}$$

where $\frac{\partial^2 \ln f_{\mathbf{X}}(\mathbf{x})}{\partial \mu_i \partial \mu_j}$ is the second-order score function which can be also analytically computed. Since the joint probability density function is known, calculation of the higher-order score function is very straightforward even for other parameters such as variance of the random variables [18]. The first and second-order sensitivity of the probability of failure in Eqs. (13) and (14) can be efficiently computed since the failure region $\hat{\Omega}_f$ is defined as the approximated performance function.

4.4. Numerical example - Reliability analysis for two dimensional performance function

Let's consider a two dimensional mathematical example given by [19]

$$g(\mathbf{X}) = \left[\exp\left(0.8X_1 - 1.2\right) + \exp\left(0.7X_2 - 0.6\right) - 5\right]/10 \tag{15}$$

where $X_1 \sim N(4, 0.8^2)$ and $X_2 \sim N(4, 0.8^2)$.

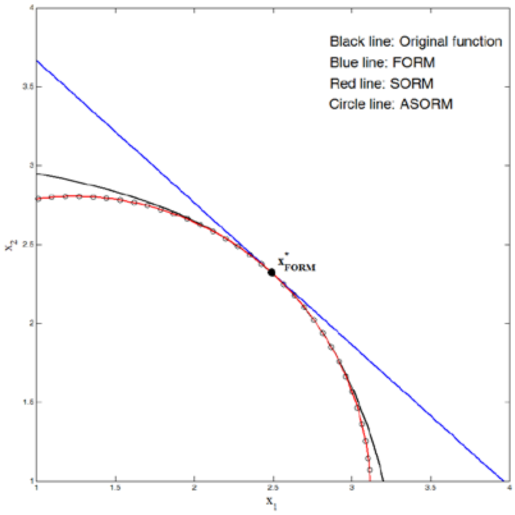

Figure 1: Limit-state equations in FORM, SORM and ASORM

Table 1: The results of reliability analysis and stochastic sensitivity analysis in the 2-D example

	FORM	SORM (Breitung [3])	Proposed SORM	MCS
P_f	0.240%	0.162%	0.153%	0.158%
$\partial P_f / \partial \mu_1$	-0.00627	-	-0.00410	-0.00424
$\partial P_f / \partial \mu_2$	-0.00695	-	-0.00455	-0.00466
$\partial^2 P_f / \partial \mu_1 \partial \mu_1$	-	-	0.00916	0.00960
$\partial^2 P_f / \partial \mu_1 \partial \mu_2$	-	-	0.0118	0.0121
$\partial^2 P_f / \partial \mu_2 \partial \mu_2$	-	-	0.0116	0.0119
F.E.	8[a]	8[a] + Hessian	8[a]	10^7

[a] The number of function evaluation and sensitivity analysis for MPP search

Figure 1 illustrates the original and approximated performance functions at MPP in X-space. The approximated performance function of ASORM is obtained using the Hessian approximated by SR1 update. Since the original performance function has a large curvature near MPP, the linearly approximated function does not sufficiently describe the nonlinearity of the original function. As illustrated in Fig. 1, since the functions in SORM and ASORM locally approximate the original function near MPP quite well, the accurate reliability can be estimated by applying the importance sample selecting MPP as a sampling center to the quadratically approximated function. Table 1 shows results of the reliability analysis. In order to obtain the reference solution, Monte Carlo simulation (MCS) with sample size of 10^7 is performed. Due to the high nonlinearity as illustrated in Fig. 1, FORM shows significant error in estimating the probability of failure as well as the stochastic sensitivity. While SORM proposed by Breitung reduces the error of the probability of failure by calculating the Hessian of the performance function, it is quite difficult to calculate the stochastic sensitivity due to the absence of research on the stochastic sensitivity in SORM. The proposed eSORM evaluates accurate probability of failure by using the approximated Hessian. Furthermore, second-order sensitivity as well as first-order one is accurately calculated. In spite of the improvement in terms of accuracy, additional function calls are not required after MPP search since the proposed method uses the approximated Hessian utilizing the previous derivative information.

5. Conclusions

An accurate and efficient reliability analysis method is proposed in this paper. By utilizing the importance sampling to estimate the probability of failure and its sensitivities, the proposed eSORM can achieve better accuracy than conventional SORM methods since the complete second-order Taylor expansion of a performance function is used. In the proposed eSORM, SR1 update is also utilized using derivatives data obtained during MPP search in order to approximate the Hessian of the performance function. In this way, eSORM requires computation only used in MPP search, indicating that the proposed method shows the same efficiency with FORM.

6. References

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