Sensitivity Analysis of Structural Response to External Load Position

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Abstract

Procedures for sensitivity analysis of the static responses of a plate or shell structure, i.e., the nodal displacement and mean compliance, with respect to the position of an external applied load are developed in this work mainly because these responses are highly affected by the application point of the load imposed. Based on the essential ideas of the finite element analysis, an external concentrated load is first transformed into the equivalent nodal forces such that the influence of its move or shift is represented completely by the value variation of the associated nodal forces. As a result, the first-order derivatives of an external load to its location change can be performed appropriately by the aid of the shape functions of a plate element. Subsequently, the relevant sensitivities of the structural responses are formulated readily with the discrete methodology upon the finite element formulation. Finally, a numerical example is provided to demonstrate the validity of the sensitivity formulations presented, and the numerical results show the high accuracy of the response sensitivity calculations.

Keywords: sensitivity analysis; external applied load location; nodal displacement; compliance;

1. Introduction

In the preliminary design stage of an engineering structure, there always exist some uncertainties associated not only with the structural design parameters, but also with a part or all of the external loads. That is, the loads imposed to the structure may often experience modifications in their values, directions and/or positions during the design process in order to testify the resulting design to comply with some strict regulations in various circumstances [1]. Over the past several decades, the mechanical analysis and optimization of a plate- or shell-type structure with design dependent or independent loadings were undertaken extensively in many aspects. Most commonly, the application position of an external load is fixed in [2, 3]. However, it is also noticed that even a small move or shift of external load may bring about a significant influence on both the performance of the structure and then the structure final design. This fact makes it highly desirable for developing an efficient technique capable of estimating variations of the structural performances due to a small motion of an applied load, and such a sensitivity analysis can quantitatively afford an explicit and quick solution to the problem.

As is widely known, the sensitivity analysis is extremely useful in the scope of design optimization processes, especially in the gradient-based optimization algorithms, where the design sensitivity can be used as a guide to implement the structural modification. In effect, the design sensitivity analysis has become a major computational cost in most structural optimization solutions [4]. As a result, a simple and precise formulation for an adequate sensitivity calculation has always been an active topic in the fields of the structural optimization design.

This paper is aimed to extend the previous study by the present author [5] for the related sensitivity analysis of the structural responses into the plate/shell flexural situation. In this work, the sensitivity formulations of the structural displacement and mean compliance are conducted, respectively, with respect to the position variations of an external concentrated load. The sensitivity analysis is still carried out with use of the finite element (FE) analysis mainly because the corresponding responses are usually obtained on the same strategy. First, the equivalent nodal forces of the external applied load are constructed by the adequate interpolation functions of the plate element. Due primarily to this direct transformation, the effect of the external load shift is fully represented by the value variation of the equivalent nodal forces. Next, the explicit formulations of the first-order derivatives of the nodal force vector to change of the application point of the external load are derived fairly with the aid of the essential features of the plate element. Later, with the above derivations the explicit formulations of the sensitivity derivatives of the nodal displacements and the compliance are developed immediately to the movement of the external load. As the structural analysis resorts most regularly to the numerical execution with FE methodology, such a derivation of the design sensitivity has an advantage of compatibility with the numerical estimations of the structural responses upon the same discrete model, and most importantly, can be applied in conjunction with an existing commercial FE analysis package. Finally, the sensitivity calculations of the responses will be

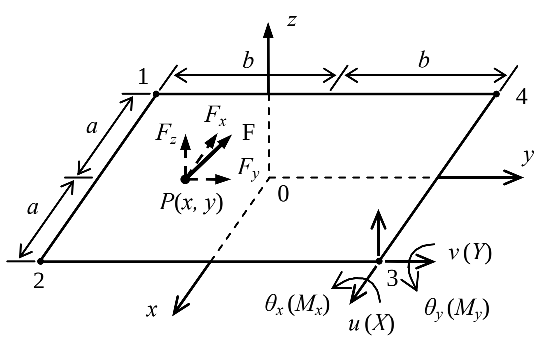

Figure 1: A rectangular plate element subject, at point $P(x, y)$, to a concentrated force **F** of the components F_x, F_y and F_z

illustrated and the resultant accuracy be demonstrated with a typical numerical example.

2. Derivative of an External Load

Consider an isotropic, thin rectangular plate (or shallow shell) element of thickness h subjected to a concentrated or point load **F** at an interior point $P(x, y)$ in the element region. Figure 1 shows a typical rectangular plate element with 5 degrees of freedom (DOFs) at each of the corners, and the corresponding nodal forces are shown in the parentheses. The element or local coordinate system is chosen such that the x-y plane coincides with the midplane of the plate. The external applied force may have three components F_x, F_y and F_z parallel, respectively, to each of the coordinate axes. Herein, the classical Kirchhoff hypothesis for thin plates applies for describing the transverse deflection. Based on the FE theory, the midplane displacements inside the element can be approximated as explicit functions of the element nodal displacements:

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = [N]\{d\}^e \tag{1}$$

where u and v are the in-plane displacement components in the x-y coordinate system, respectively, and w is the lateral displacement component along the z-axis at any point on the plate element midplane (i.e., z=0). $\{d\}^e$ is a vector of 20 entries denoting the generalized element nodal displacements:

$$\{d\}^e = [\{d_1\}^T \quad \{d_2\}^T \quad \{d_3\}^T \quad \{d_4\}^T]^T \tag{2}$$

in which, the superscript T denotes matrix transpose. $\{d_i\}$ represents the DOFs at each corner node i (i=1, 2, 3, 4) , see Figure 1,

$$\{d_i\} = \begin{Bmatrix} u_i \\ v_i \\ w_i \\ \theta_{xi} \\ \theta_{yi} \end{Bmatrix} = \begin{bmatrix} u_i & v_i & w_i & \left(\dfrac{\partial w}{\partial y}\right)_i & -\left(\dfrac{\partial w}{\partial x}\right)_i \end{bmatrix}^T \tag{3}$$

where u_i, v_i and w_i are the nodal displacements along each of the local axes at a node. θ_{xi} and θ_{yi} denote the associated rotations about the x- and y-axis, respectively. In the small deflection theory of thin plates, the transverse displacement w is uncoupled from the in-plane displacements u and v. Consequently, these displacements can be interpolated separately as is already known [6]. The matrix of shape functions of the plate element in Eq. (1) is

$$[N] = \big[[N_1] \quad [N_2] \quad [N_3] \quad [N_4]\big] \tag{4a}$$

in which

$$[N_i] = \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & \overline{N}_i & \overline{N}_{xi} & \overline{N}_{yi} \end{bmatrix} \quad (i = 1, 2, 3, 4) \tag{4b}$$

is the matrix of shape functions to each of the four nodes. The usual shape functions read commonly as [7]:

$$N_i = (1+\xi_i\xi)(1+\eta_i\eta)/4 \tag{5}$$

For the in-plane components u_i and v_i identically, and

$$\begin{cases} \overline{N}_i = (1+\xi_i\xi)(1+\eta_i\eta)(2+\xi_i\xi+\eta_i\eta-\xi^2-\eta^2)/8 \\ \overline{N}_{xi} = -b\eta_i(1+\xi_i\xi)(1+\eta_i\eta)(1-\eta^2)/8 \\ \overline{N}_{yi} = a\xi_i(1+\xi_i\xi)(1+\eta_i\eta)(1-\xi^2)/8 \end{cases} \tag{6a}$$

for the transverse components w_i, θ_{xi} and θ_{yi}, respectively. In the above expressions,

$$\begin{cases} \xi = \dfrac{x}{a} \in [-1, 1], \quad \xi_i = \dfrac{x_i}{a} \\ \eta = \dfrac{y}{b} \in [-1, 1], \quad \eta_i = \dfrac{y_i}{b} \end{cases} \tag{6b}$$

For a concentrated force **F** acting at a point $P(x, y)$ within an element, the equivalent nodal forces acting on the same element is defined by the virtual work principle of the external load [6],

$$\{f\}^e = [N]^T_{(x,y)} \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \tag{7}$$

Upon this simple mathematical transformation, a concentrated or point force **F**, positioned at $P(x, y)$, is now replaced entirely by the equivalent nodal force vector $\{f\}^e$ of size 20, of which the component values depend only upon the coordinates of the application point P. In other words, the outcomes induced by continuous movement of the external load **F** can now be represented completely by those resulting from the magnitude variation of the equivalent coupled nodal forces $\{f\}^e$. Therefore, the derivative research of the structural responses to the motion of the external force **F** itself turns, from now on, into the derivative analyses with respect to the relevant changes of the equivalent nodal force vector $\{f\}^e$. Moreover, the derivative of the equivalent nodal force due primarily to modification of the acting point of the external load can be performed with ease,

$$\frac{\partial\{f\}^e}{\partial x} = \frac{\partial[N]^T}{\partial x}\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix}, \quad \frac{\partial\{f\}^e}{\partial y} = \frac{\partial[N]^T}{\partial y}\begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} \tag{8}$$

3. Derivatives of Nodal Displacements of a Structure

Suppose a general plate structure is modeled with an adequate FE mesh, the overall equations of the force equilibrium in terms of the nodal displacements are represented as follows:

$$[K]\{u\} = \{P\} + \{F(s)\} \tag{9}$$

where $[K]$ is the system stiffness matrix, $\{u\}$ is a vector of the unknown nodal displacements, which is dependent on both the magnitude and position of the external loads. On the right hand of the equation, the external loads applied on the structure are separated into two parts: $\{P\}$ is the load invariable during the design process, whereas $\{F(s)\}$ is a vector of the nodal forces which is a function of the application position variable s. Taking partial derivatives of Eq. (9) with respect to a location variable s of the external applied load yields:

$$[K]\frac{\partial\{u\}}{\partial s} = \frac{\partial\{F(s)\}}{\partial s} \tag{10}$$

here it has been supposed that the plate structure itself is essentially independent of the external applied loads. Notably, the procedure for the displacement sensitivity solution $\partial\{u\}/\partial s$ can be carried out by solving Eq. (10) caused by $\partial\{F(s)\}/\partial s$, which is rather similar to the solution of the displacement $\{u\}$ under the external loads, see Eq. (3). In view of this similarity, $\partial\{F(s)\}/\partial s$ is generally considered as a pseudo or fictitious load for each of the position variables [5].With the relevant result attained in Eq. (8), by assembly of the related element results, it is then a trivial task to computed the first-order sensitivity of the nodal displacement vector of a plate structure,

$$\frac{\partial\{u\}}{\partial s} = [K]^{-1}\frac{\partial\{F(s)\}}{\partial s} \tag{11}$$

4. Compliance Sensitivity Formulation

In the past years, the topology optimization problems of a plate structure with consideration of the overall stiffness have been investigated exhaustively, where the structure is generally designed to be stiff enough to carry a given set of the external loads properly [2, 3, 8]. Usually the mean compliance, i.e., the total work done by the external loads under a particular configuration, is posed as the objective function. Thus, it is recognized that the compliance is actually a highly load-dependent measurement of the structural stiffness in this capacity. Consequently, the sensitivity analysis of the compliance to the external applied force is of particular interest in practical cases since a small movement of the external load may remarkably alter the system compliance obtained.

The mean compliance C of a plate structure is defined as half the scalar product of the applied forces and the corresponding displacements at equilibrium, and is given in the FE format:

$$C = \frac{1}{2}\left(\{P\} + \{F(s)\}\right)^T\{u\} = \frac{1}{2}\left(\{P\} + \{F(s)\}\right)^T[K]^{-1}\left(\{P\} + \{F(s)\}\right) \tag{12}$$

Differentiating the above expression with respect to the location variable s yields

$$\frac{\partial C}{\partial s} = \left(\{P\} + \{F(s)\}\right)^T[K]^{-1}\frac{\partial\{F(s)\}}{\partial s} = \{u\}^T\frac{\partial\{F(s)\}}{\partial s} = \sum_{e=1}^{n_e}\left(\{d\}^e\right)^T\frac{\partial\{f\}^e}{\partial s} \tag{13}$$

Obviously, the compliance sensitivity can be computed as long as the displacements and the derivative of the external applied load, or the pseudo load, are obtainable. With this sensitivity result, it is capable to make a primary estimate of the structural compliance value after variation of the load position by the linear Taylor series expansion,

$$\overline{C} \approx C + \frac{\partial C}{\partial s} \Delta s \tag{14}$$

5. Illustrative Example

In this section, the numerical results of the structural responses are solved using the FE analyzer ANSYS program. Then, the corresponding sensitivities are evaluated with the program coded in Matlab providing several digits of accuracy to verify the reliability and accuracy of the sensitivity formulation presented.

A quarter cylinder shell of the thickness h=2 mm is simply supported at its four corners while it bears a concentrated vertical point force F=3 kN at the structural center. Figure 2 displays the geometry, dimensions and external load. The structural projection onto the x-y plane is a square. At the same time, the gravity load is also involved in the analysis (the gravitational constant g= 9.8 m/s^2). The cylinder is discretized with a sufficiently fine mesh of the finite elements 18 × 18. Assume the Young's modulus is E=210 GPa, Poisson ratio v=0.3 and the mass density ρ=2800 kg/m^3. First, the displacement values and the related sensitivities at the center points of the free edges, Point A and B, are computed to the position move of the vertical load in the x- and y-axis, respectively, and indicated in Table 1.

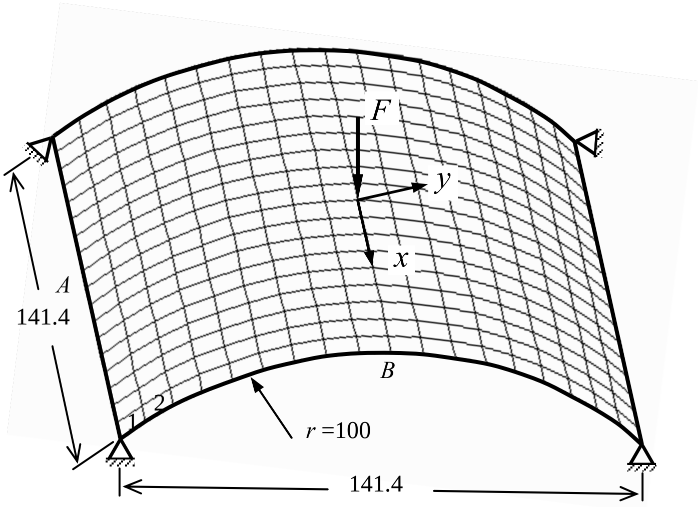

Figure 2: A quarter cylinder simply supported at corners is subject to a vertical concentrated force F at the center point

Table 1: Displacements at points A and B and the corresponding sensitivities to the position of the external vertical forces applied at the center of the cylinder shell

Point	Displacements		Sensitivity to position of the vertical force in the axis	
	component	value	x	y
A	u (mm)	0	4.87261E-5	0
	v (mm)	-4.67423E-1	7.38329E-7	-1.09879E-2
	w (mm)	5.68054E-1	-1.25806E-6	1.47863E-2
	θ_x (rad)	-1.36695E-2	1.54653E-5	4.77687E-1
	θ_y (rad)	0	-9.91461E-2	0
	θ_z (rad)	0	-9.08507E-2	0
B	u (mm)	-9.19519E-3	1.03708E-3	-1.07670E-6
	v (mm)	0	0	-1.64534E-2
	w (mm)	-3.69736E-1	-8.65907E-3	3.24734E-6
	θ_x (rad)	0	0	-1.15354
	θ_y (rad)	2.60914E-4	1.22743E-1	-1.70148E-4
	θ_z (rad)	0	0	-3.95777E-3

First of all, it can be seen that the displacement terms u, θ_y and θ_z at Point A and the corresponding sensitivities to the y-motion of the concentrated load are all zeroes. So are the terms v, θ_x and θ_z at Point B to its x-motion. This is true due essentially to the symmetry of the structure. It is therefore understood that these displacements are very blunt to the position disturbances of the applied force in the related directions. However, it is found that the sensitivity to the companion axial move is not null, and the results are comparatively very notable. Secondly, the related sensitivities are much smaller for the major displacement terms *v*, *w* and θ_x at Point *A* (in gray) to the *x*-motion of the concentrated load than to its *y*-motion. A similar observation can also be made about the sensitivities of the displacement terms *u*, *w* and θ_y at Point *B* (in gray) to the *y*-motion of the vertical load than to its *x*-motion. This is just the case as is expected.

Table 2: Displacements at points *A* and *B* before and after an imaginary move of the external vertical force individually in the *x*- or *y*-directions by an element size

Point	component	Displacements with the external force moved by an element size in the positive direction of different axes: linear estimation *x*	linear estimation *y*	solution by FEM *x*	solution by FEM *y*
A	*u* (mm)	3.82829E-4	0	3.47619E-4	0
	v (mm)	-4.67417E-1	-5.63189E-1	-4.64374E-1	-5.43768E-1
	w (mm)	5.68044E-1	6.96925E-1	5.62859E-1	6.68839E-1
	θ_x (rad)	-1.36694E-2	-9.50617E-3	-1.36056E-2	-9.30330E-3
	θ_y (rad)	-7.78966E-4	0	-7.72402E-4	0
	θ_z (rad)	-7.13790E-4	0	-7.07776E-4	0
B	*u* (mm)	-1.04714E-3	-9.20457E-3	-7.93013E-4	-8.24312E-3
	v (mm)	0	-1.43401E-1	0	-1.41692E-1
	w (mm)	-4.37768E-1	-3.69708E-1	-4.41272E-1	-3.49618E-1
	θ_x (rad)	0	-1.00539E-2	0	-9.91328E-3
	θ_y (rad)	1.22528E-3	2.59431E-4	1.30799E-3	3.99758E-4
	θ_z (rad)	0	-3.44943E-5	0	-3.38898E-5

Based on the displacement sensitivities shown in Table 1, it is quite simple to predict the nodal displacement terms with a conceived motion of the concentrated force *F* in *x*- or *y*-direction. The new displacements $\{\bar{d}\}$ after a position perturbation of the concentrated load by a small step can be estimated in the linear Taylor series

$$\{\bar{d}\} = \{d\} + \frac{\partial\{d\}}{\partial s}\Delta s \tag{15}$$

Table 2 lists the deflections at Point *A* and *B* with the position of the vertical concentrated force disturbed by an element size individually along the axes. In contrast, the corresponding solutions with the FE method are given simultaneously for comparison. By a close examination, it is found that the displacements estimated linearly on the sensitivity analysis are in good agreement with the FE solution for the major displacement terms.

Table 3: Structural mean compliances and the related sensitivities to the position of the external vertical load applied at the center of the cylinder shell

Compliance (J)	First-order sensitivity in different directions: axis	value	Compliance with the concentrated load moved by an element size in the positive direction: estimation	by FEM
1.06775	*x*	4.92985E-2	1.06814	1.07465
	y	1.94058E-1	1.06944	1.10675

The structural mean compliance and the associated first-order sensitivities to the position of the vertical load are illustrated in Table 3. It is shown evidently from the sensitivities that the structural compliance will increase if the vertical force moves away from the center. Therefore, it can be known explicitly that the center of the cylinder in the present example is the most appropriate location for the vertical load under the particular consideration of the structural stiffness. Moreover, it is observable from Table 3 that the compliance is much more sensible to the *y*-direction move than to its *x*-direction move of the vertical load. This is very important for the analysis of the uncertain external forces in practice.

6. Conclusions

In the preliminary design of a practical structure, the external applied loads may change their application locations in order to testify that the structural design is appropriate to support or withstand the loads in various conditions. The ability to quickly evaluate the response changes of a structure to the location variation of an external load is of great importance to the structural designers/analysts. This paper is really an extension of the previous work by the author for 2D plane stress conditions [5]. By means of the discrete method in combination with the features of the plate/shell element, the derivative of an external load is first obtained. Then, the sensitivity expressions of the structural responses, such as the nodal displacement and mean compliance, are developed readily for a bending plate structure. Subsequently, a typical example is presented to validate the sensitivity formulations, and the numerical results show that the proposed process can provide the response sensitivities with an excellent accuracy.

Acknowledgements

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