11th World Congress on Structural and Multidisciplinary Optimisation
7th -12th, June 2015, Sydney Australia

Accuracy Improvement of MPP-Based Dimension Reduction Method Using the Eigenvectors of the Hessian Matrix

Seongbin Kang[1], Ikjin Lee[1], Jongmin Lim[1]

[1] Korea Advanced Institute of Science and Technology, Daejeon, Korea, kangsb@kaist.ac.kr

1. Abstract
The main purpose of this study is to develop an accurate methodology for the most probable point (MPP)-based dimension reduction method (DRM) by proposing a proper orthogonal transformation to calculate a probability of failure more accurately. In this study, dependency of an axis direction is shown in the univariate DRM, indicating that the probability of failure can be differently calculated according to a different orthogonal transformation. In order to obtain a proper axis direction for DRM, the Hessian of a performance function is utilized in this study. By performing orthogonal transformation using eigenvectors of the Hessian matrix, axes of Gaussian quadrature points for numerical integration are selected along the principal eigenvector directions of the Hessian. In this way, the error incurred by the univariate dimension reduction is minimized, so the probability of failure can be calculated more accurately. Numerical examples verify the accuracy of the proposed method by comparing with existing MPP-based DRM.

2. Keywords: Dimension Reduction Method (DRM), Most Probable Point (MPP), First-Order Reliability Method (FORM), Hessian Matrix, Eigenvector

3. Introduction
Nowadays engineers are facing new and emerging challenges which include intensive use of computational simulations and application of new technologies into complex systems. Due to this, the requirements of high quality and reliability, and the reliable decision-making under uncertainty are essential. To overcome these things, reliability analysis has been advanced gradually. As a result, reliability analysis has been widely and successfully applied to various engineering applications.

There are various reliability analysis methods and the most popular methods are analytical methods and sampling methods. Analytical methods are called most probable point(MPP)-based methods which include First-Order Reliability Method(FORM) [1,2], Second-Order Reliability Method(SORM) [3-5] and Dimension Reduction Method(DRM) [6-10]. When calculating the probability of failure by FORM and SORM, a performance function is approximated by the first or second-order Taylor series expansion at MPP. Through a linear approximation, the probability of failure could be simply calculated in FORM. However, if the performance function is highly nonlinear and/or multidimensional, the result of FORM could be erroneous. SORM is definitely more accurate than FORM because it approximates the performance function in a quadratic form. But SORM also includes errors [5] caused by a few approximations. MPP-based DRM [8-10] has been recently proposed to approximate a multidimensional function using the sum of lower dimensional functions. MPP-based DRM is much more accurate than FORM and users can control its accuracy by changing the number of quadrature (or integration) points. In the MPP-based DRM, probability of failure calculation is related to axis directions because the quadrature points are located along the axes.

The main objective of this study is to improve the accuracy of the MPP-based DRM using eigenvectors of the Hessian matrix. The probability of failure calculation in the MPP-based DRM changes according to the axes because locations of quadrature points depend on the axes. To obtain more accurate calculation, the quadrature points should be located in proper position. In other words, the proper axes should be obtained through an appropriate transformation from the original space. Therefore it is a main issue to find an orthonormal rotation matrix which arranges axes in the proper position in the proposed method. To apply the proposed method, an MPP should be found first after transforming all random variables in the original **X**-space to the standard normal **U**-space through the Rosenblatt transformation [12]. After finding MPP, the existing method (i.e., MPP-based DRM) uses Gram-Schmidt orthogonalization to obtain the rotation matrix but the proposed method has additional process which uses eigenvectors of the Hessian matrix at MPP in **U**-space. With the rotation matrix which is obtained by using eigenvectors of the Hessian matrix, the quadrature points are arranged in proper position and the probability of failure calculation becomes more accurate.

The paper is organized as follows. Section 4 covers basic concepts of reliability analysis. Section 5 covers the detail process of the proposed method. In Section 6, numerical examples are tested to demonstrate that the proposed MPP-based DRM can calculate the probability of failure of a performance function more accurately than the existing MPP-based DRM.

4. Basic concepts and review for previous method

4.1 FORM and SORM

FORM has been extensively used for a reliability analysis which includes calculation of probability of failure, denoted as P_F which is defined using a multidimensional integral [1, 11]

$$P_F \equiv P[G(\mathbf{X}) > 0] = \int_{G(\mathbf{X})>0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{1}$$

where $G(\mathbf{X})$ is the performance function such that $G(\mathbf{X}) > 0$ is defined as failure. $\mathbf{X} = \{X_1, X_2, \quad, X_N\}^{\mathrm{T}}$ is an N-dimensional random vector where the upper case X_i means that they are random variables, and $f_{\mathbf{X}}(\mathbf{x})$ is a joint PDF of $\mathbf{X}$.

For calculation of the probability of failure in Eq. (1), FORM linearizes $G(\mathbf{X})$ at MPP in $\mathbf{U}$-space obtained by the Rosenblatt transformation [12] and first-order Taylor approximation such that

$$G(\mathbf{X}) = g(\mathbf{U}) \cong g_L(\mathbf{U}) = g(\mathbf{u}^*) + \nabla g^{\mathrm{T}}(\mathbf{U} - \mathbf{u}^*) \tag{2}$$

where $\mathbf{u}^*$ is the MPP in $\mathbf{U}$-space which means the minimum distance point on the limit state function from the origin. MPP can be found by solving the following optimization problem to

$$\begin{aligned} &\text{minimize} && \|\mathbf{u}\| \\ &\text{subject to} && g(\mathbf{u}) = 0 \end{aligned} \tag{3}$$

∇g is the gradient vector of the performance function calculated at the MPP in $\mathbf{U}$-space. A reliability index, denoted as β, is defined as the distance from the origin to $\mathbf{u}^*$ [2]. With a linearized performance function and the reliability index β, FORM approximates the probability of failure in Eq. (1) as

$$P_F^{\mathrm{FORM}} \cong \Phi(-\beta) \tag{4}$$

where $\Phi(\bullet)$ is the standard normal cumulative distribution function (CDF).

Using a normalized MPP vector $\boldsymbol{\alpha}$, quadratic approximation of the performance function in $\mathbf{U}$-space and rotational transformation, the probability of failure can be approximated by SORM as [3, 4, 9]

$$P_F^{\mathrm{SORM}} \cong \Phi(-\beta)\left|\mathbf{I}_{N-1} - 2\frac{\phi(\beta)}{\Phi(-\beta)}\mathbf{A}_{N-1}\right|^{-\frac{1}{2}} \tag{5}$$

where $\mathbf{A} = \begin{bmatrix} \mathbf{A} & \mathbf{A}_{1\mathrm{N}} \\ \mathbf{A}_{\mathrm{N}1} & \mathbf{A}_{\mathrm{NN}} \end{bmatrix} = \dfrac{\mathbf{R}^{\mathrm{T}}\mathbf{H}\mathbf{R}}{2\|\nabla g\|}$, $\mathbf{H}$ is the Hessian matrix of the performance function at MPP, and $\mathbf{R}$ is the orthonormal rotation matrix used in $\mathbf{U} = \mathbf{R}\mathbf{V}$.

4.2 MPP-based DRM

DRM is a method to approximate the multi-dimensional integration of the performance function using a function with reduced dimension. There are several types of DRM according to the level of dimension reduction. In MPP-based DRM which uses a univariate DRM, any N-dimensional performance function $G(\mathbf{X})$ can be reduced to summation of one-dimensional functions as [8, 9]

$$G(\mathbf{X}) \cong \hat{G}(\mathbf{X}) \equiv \sum_{i=1}^{N} G(x_1^*, \quad, x_{i-1}^*, X_i, x_{i+1}^*, \quad, x_N^*) - (N-1)G(\mathbf{x}^*) \tag{6}$$

where $\mathbf{x}^* = \{x_1^*, x_2^*, \quad, x_N^*\}^{\mathrm{T}}$ is the MPP obtained by Eq. (3). To calculate the probability of failure using the MPP-based DRM, transformation to rotated standard normal $\mathbf{V}$-space is required. To rotate $\mathbf{U}$-space into $\mathbf{V}$-space, it is necessary to construct an **N×N** orthonormal matrix **R** whose N$^{\text{th}}$ column is the normalized MPP vector $\boldsymbol{\alpha}$, i.e., $\mathbf{R} = [\mathbf{R_1} | \, \boldsymbol{\alpha}]$ where N×(N-1) matrix $\mathbf{R_1}$ satisfies $\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{R_1} = \mathbf{0}$ and N is the number of random variables. In **V**-space, the probability of failure can be expressed as

$$P_F \equiv P[G(\mathbf{V}) > 0] = \int_{G(\mathbf{V})>0} f_{\mathbf{V}}(\mathbf{v})\mathrm{d}\mathbf{v} \tag{7}$$

where $G(\mathbf{V}) = G(\mathbf{X}(\mathbf{V}))$ is obtained from the Rosenblatt transformation and rotational transformation. The probability of failure is calculated using the constrain shift such that [10]

$$G^s(\mathbf{v}) \equiv G(\mathbf{v}) - G(\mathbf{v}^*) \tag{8}$$

where $G^S(\mathbf{v})$ is a shifted performance function and $\mathbf{v}^* = \{0, \quad , 0, \beta\}^{\mathrm{T}}$ is the MPP in **V**-space. In a general N-dimensional space, $G^S(\mathbf{v})$ is expressed as [10]

$$\begin{aligned} G^S(\mathbf{v}) \cong \hat{G}^S(\mathbf{v}) &\equiv \sum_{i=1}^{N} G_i^S(v_i) - (N-1)G^S(\mathbf{v}^*) \\ &= G^S(v_1, v_2^*, \quad , v_N^*) + G^S(v_1^*, v_2, \quad , v_N^*) + \quad + G^S(v_1^*, v_2^*, \quad , v_N) \\ &= G^S(v_1, 0, \quad , \beta) + G^S(0, v_2 \quad , \beta) + \quad + G^S(0, 0, \quad , v_N) \end{aligned} \tag{9}$$

Using the first-order Taylor series expansion at the MPP, the last univariate component in Eq. (9) is linearly approximated as

$$G^S(0, \quad , v_N) \cong \left.\frac{\partial G(\mathbf{v})}{\partial v_N}\right|_{\mathbf{v}=\mathbf{v}^*} (v_N - \beta) = b_1(v_N - \beta), \;\; b_1 = \left.\frac{\partial G(\mathbf{v})}{\partial v_N}\right|_{\mathbf{v}=\mathbf{v}^*} \tag{10}$$

This linear approximation along v_N -axis is also used for the probability of failure calculation in SORM [3, 4]. Using the above equations and the linear approximation, the probability of failure in the MPP-based DRM is calculated as [9, 10, 13, 14]

$$P_{\mathrm{F}}^{\mathrm{DRM}} \cong \frac{\prod_{i=1}^{N-1} \sum_{j=1}^{n} w_j \Phi\left(-\beta + \frac{G_i^S(v_i^j)}{b_1}\right)}{\Phi(-\beta)^{N-2}} \tag{11}$$

where v_i^j are quadrature points, w_j are weights, and n is the number of quadrature points and weights. Table 1 shows values of the quadrature points and weights.

Table 1. Gaussian quadrature points and weights

n	Quadrature points	Weights
1	0.0	1.0
3	$\pm\sqrt{3}$	0.166667
	0.0	0.666667
5	± 2.856970	0.011257
	± 1.355626	0.222076
	0.0	0.533333

5. MPP-based DRM using the eigenvectors of the Hessian matrix

The objective of the MPP-based DRM using eigenvectors of the Hessian matrix is to improve the accuracy of DRM. Dependency of an axis direction is shown in the univariate DRM, indicating that the probability of failure would be calculated differently according to a different orthogonal transformation. In order to obtain a proper axis direction for DRM, eigenvectors of the Hessian matrix of a performance function at MPP is used when transforming **U**-space to **V**-space. This method consists of 2 steps to obtain the rotation matrix that has the Hessian information before calculating the probability of failure by the MPP-based DRM

Step 1. Transforming **U**-space to **H**-space using eigenvectors of the Hessian

First, it is required to find eigenvectors of the Hessian matrix of a performance function $g(\mathbf{U})$ at MPP. Eigenvectors are arranged in column vectors of the first rotation matrix $\mathbf{R_1}$. Using the orthonormal transformation $\mathbf{U} = \mathbf{R_1 H}$, **U**-space is transformed to the standard normal **H**-space. Through this process, all axes are arranged in eigenvector directions of the Hessian of $g(\mathbf{U})$ and thus cross-term effects can be minimized.

Step 2. Transforming **H**-space to **V**-space

To apply the MPP-based DRM, $\mathrm{N^{th}}$ axis direction should be aligned to the $\boldsymbol{\alpha}$-direction which is the normalized MPP vector in **U**-space. To obtain this axis, it is necessary to obtain an $\mathrm{N \times N}$ orthonormal matrix $\mathbf{R_2}$ whose $\mathrm{N^{th}}$ column is the normalized MPP vector in **H**-space given by

$$\tilde{\boldsymbol{\alpha}} \equiv \mathbf{R}_1^{-1} \boldsymbol{\alpha} \tag{12}$$

i.e., $\mathbf{R_2} = [\mathbf{R_3} | \, \tilde{\boldsymbol{\alpha}}]$ where $\mathrm{N \times (N-1)}$ matrix $\mathbf{R}_3$ satisfies $\tilde{\boldsymbol{\alpha}}^{\mathrm{T}} \mathbf{R}_3 = \mathbf{0}$ and N is the number of random variables. $\mathbf{R}_3$ is determined by the Gram-Schmidt orthogonalization. The Gram-Schmidt process takes linearly independent set $\mathrm{S} = \{s_1, \cdots, s_N\}$ which consists of the normalized MPP vector($s_1 = \tilde{\boldsymbol{\alpha}}$) and standard basis vector($s_2, \cdots s_N$) and it generates an orthogonal set $\mathrm{S}' = \{s_1', \cdots, s_N'\}$. Normalized S' vector set comprises an orthonormal matrix $\mathbf{R_2}$. Using the orthonormal transformation $\mathbf{H} = \mathbf{R_2 V}$, **H**-space is rotated to the standard normal **V**-space. Figure 1 shows 3-D limit state function contours in each space obtained from the proposed method.

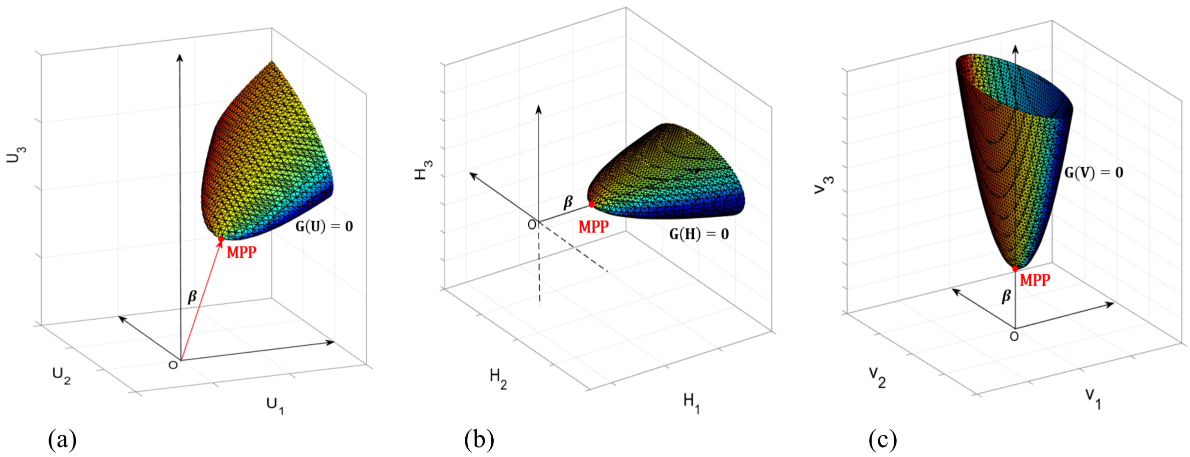

Fig. 1. 3-D limit-state function contour in (a) **U**-space; (b) **H**-space; (c) **V**-space

Through these 2 steps, 2 orthonormal transformations are combined into $\mathbf{U} = \mathbf{R_1 R_2 V}$. Substituting $\mathbf{R_1 R_2}$ with $\mathbf{R}_\mathrm{H}$, **U**-space is rotated(transformed) to **V**-space by $\mathbf{R}_\mathrm{H}$. Finally, the orthonormal rotation matrix $\mathbf{R}_\mathrm{H}$ is obtained which has the Hessian information of $g(\mathbf{U})$ and the $\mathrm{N^{th}}$ column of $\mathbf{R}_\mathrm{H}$ is the normalized MPP vector $\boldsymbol{\alpha}$. Using $\mathbf{R}_\mathrm{H}$, **V**-space axes have the Hessian information and quadrature points required to calculate the probability of failure are aligned in the **V**-space axes direction. For this reason, accuracy of the proposed MPP-based DRM is further improved. Figure 2 shows the position of the quadrature points of the limit state function contour in $\mathrm{V_1} - \mathrm{V_2}$ plane depending on whether the eigenvector direction is used or not.

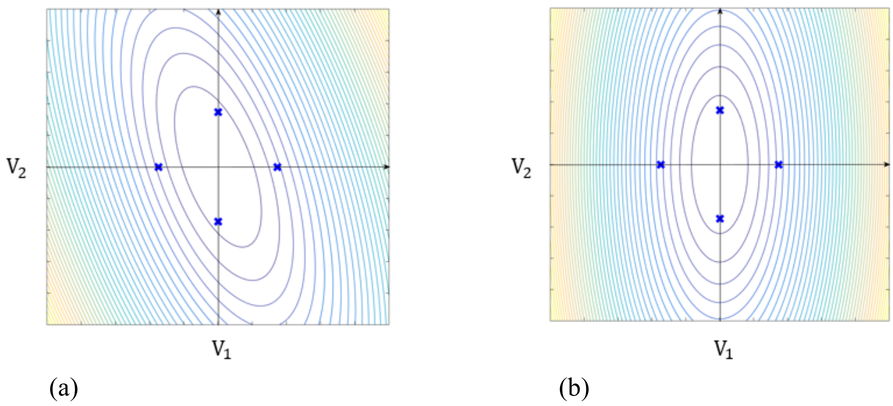

Fig. 2. $V_1 - V_2$ plane contour : (a) MPP-based DRM; (b) Proposed DRM

6. Numerical example

Accuracy improvement of the MPP-based DRM using eigenvectors of the Hessian matrix is verified by comparing it with the existing MPP-based DRM. To compare 2 methods, consider a 3-dimensionl high-order function given by

$$G_1(\mathbf{X}) = X_1^4 + X_2^3 + X_3 + X_1X_2^2 + X_2X_3 - 12 \tag{11}$$

where the random variable X_i ($i = 1, 2, 3$) follows the standard normal distribution. For the comparison, Monte Carlo Simulation (MCS) [15] result is considered as the true value of the probability of failure. To calculate the probability of failure, 3 quadrature points along each axis are used. The calculation of the probability of failure in the proposed DRM has less error compared to MCS result than existing DRM as shown in Table 2. The effect of the proposed method is shown visibly in Fig. 3 which includes the location of quadrature points and the contours of the performance function in $V_1 - V_2$ plane which are obtained by the proposed method and existing method, respectively. The contours show that the proposed DRM minimizes the cross-term effect of the performance function.

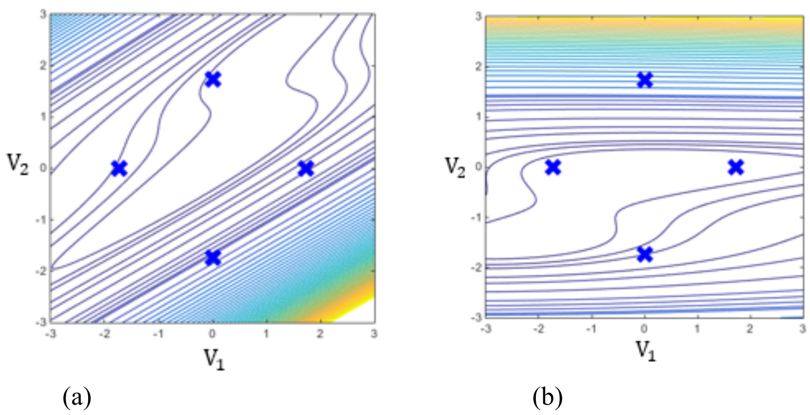

Fig. 3. $V_1 - V_2$ plane contour in 3-D example : (a) MPP-based DRM; (b) Proposed DRM

Table 2. Probability of failure calculation by two methods for 3-D example

	MPP-based DRM (3pts)	Proposed DRM (3pts)	MCS (10^7 samples)
P_F (%)	4.14	6.50	7.90
Error (%)	47.59	17.7	-

7. Conclusion

To improve the accuracy of the existing MPP-based DRM, it is proposed to use eigenvectors of the Hessian matrix when transforming to the rotated standard normal **V**-space. To verify the accuracy improvement in calculation of the probability of failure, a 3-D example with high order is tested and the results show that using the proposed MPP-based DRM is more accurate than the existing method. More research on the proposed MPP-based DRM and application to Reliability Based Design Optimization (RBDO) problems are ongoing.

8. Acknowledgements

This work was supported by the Industrial Innovation Technology Development Program in Plant Engineering Area (10048341) funded by the Ministry of Trade, Industry & Energy (MOTIE, Korea).

9. References

[1] Haldar, and S. Mahadevan, Reliability and Statistical Methods in Engineering Design, *John Wiley & Sons,* New York, NU, 2000.

[2] A.M. Hasofer, and N.C. Lind, An exact and invariant first order reliability format*, ASCE Journal of the Engineering Mechanics Division*, 100 (1), 111-121, 1974.

[3] M. Hohenbichler, and R. Rackwits, Improvement of second-order reliability estimates by importance sampling, *ASCE Journal of Engineering Mechanics*, 114(12), 2195-2199, 1988.

[4] K. Breitung, Asymptotic approximations for multinormal integrals, *ASCE Journal of Engineering Mechanics*, 110 (3), 357-366, 1984.

[5] S. Adhikari, Reliability Analysis Using Parabolic Failure Surface Approximation, *ASCE J. Eng. Mech*., 130 (12), 1407–1427, 2004.

[6] S. Rahman, and H. Xu, A univariate dimension–reduction method for multidimensional integration in stochastic mechanics, *Probabilistic Engineering Mechanics*, 19, 393–408, 2004.

[7] H. Xu, and S. Rahman, A generalized dimension–reduction method for multidimensional integration in stochastic mechanics, *International Journal for Numerical Methods in Engineering,* 61 (12), 1992–2019, 2004.

[8] S. Rahman, D. Wei, A univariate approximation at most probable point for higher-order reliability analysis, *International Journal of Solids and Structures*, 43, 2820–2839, 2006.

[9] D. Wei, A univariate decomposition method for higher-order reliability analysis and design optimization, Ph.D. Thesis, University of Iowa, 2006.

[10] Lee, K. K. Choi, L. Du, and D. Gorsich, Inverse Analysis Method Using MPP-Based Dimension Reduction for Reliability-Based Design Optimization of Nonlinear and Multi-Dimensional Systems, *Comput. Methods Appl. Mech. Eng*., 198 (1), 14–27, 2008.

[11] H. O. Madsen, S. Krenk, and N. C. Lind, Methods of Structural Safety, *Prentice-Hall, Englewood Cliffs*, NJ, 1986.

[12] M. Rosenblatt, Remarks on a Multivariate Transformation, *Ann. Math. Stat.*, 23, 470–472, 1952.

[13] H. Xu, and S. Rahman, A moment-based stochastic method for response moment and reliability analysis, *Proceedings of 2nd MIT Conference on Computational Fluid and Solid Mechanics*, Cambridge, MA, 2003.

[14] K.E. Atkinson, An Introduction to Numerical Analysis, *John Wiley & Sons*, New York, NY, Chapter 5, 1989.

[15] C.Y. Lin, W.H. Huang, M.C. Jeng, J.L. Doong, Study of an assembly tolerance allocation model based on Monte Carlo simulation, *Journal of Materials Processing Technology,* 70, 9–16, 1997.