11th World Congress on Structural and Multidisciplinary Optimisation
07th -12th, June 2015, Sydney Australia

Utilization of Gaussian Kernel Reliability Analyses in the Gradient-based Transformed Space for Design Optimization with Arbitrarily Distributed Design Uncertainties

Po Ting Lin[1]

[1] Department of Mechanical Engineering, Chung Yuan Christian University, Chungli, Taoyuan, Taiwan,
potinglin@cycu.edu.tw

1. Abstract
Several Reliability-Based Design Optimization (RBDO) algorithms have been developed to solve engineering optimization problems under design uncertainties. Some existing methods transform the random design space to standard normal design space to estimate the reliability assessment for the evaluation of the failure probability. When the random variable is arbitrarily distributed and cannot be properly fitted to any known form of probability density function, the existing RBDO methods, however, cannot perform reliability analysis either in the original design space or in the standard normal space. This paper proposes a novel method, Ensemble of Gradient-based Transformed Reliability Analyses (EGTRA), to evaluate the failure probability of arbitrarily distributed random variables in the original design space. The arbitrary distribution of the random variable is approximated by a merger of multiple Gaussian kernel functions. Each Gaussian kernel function is transformed to a single-variate coordinate that is directed toward the gradient of the constraint function. The failure probability is then estimated by the ensemble of each kernel reliability analysis. This paper further derives a linearly approximated probabilistic constraint at the design point with allowable reliability level in the original design space using the aforementioned fundamentals and techniques. Numerical examples with generated random distributions show EGTRA is capable of solving the RBDO problems with arbitrarily distributed uncertainties in the original design space.
2. Keywords: gradient-based transformation; Gaussian kernel density estimation; reliability-based design optimization; arbitrarily distributed design uncertainty.

3. Introduction
In engineering design, traditional deterministic optimization has been successfully applied to improve quality, processes and reduced costs. However, uncertainties have to be considered to make designs more confident and reliable. These traditional deterministic approaches for optimization of components, products and systems are slowly being replaced in the past decades of approaches that integrate probabilistic considerations. These probabilistic considerations were investigated and have been known to be material property deviation [1], allowable failure probabilities from standards [2], production condition [3], reported incidences of failures or satisfactions [4] and operating conditions [5], to mention a few. The basic concept behind Reliability-Based Design Optimization (RBDO) methods is to integrate and consider these probabilistic factors in the optimization process and many approaches have been developed in the past.
Reliability Index Approach (RIA) [6-9] formulated the probabilistic constraints based on the evaluations of reliability indices. Lin et al. [10] resolved the convergence problems of RIA [11] by modifying the reliability index to correctly evaluate the failure probability for both feasible and infeasible design points. Performance Measure Approach (PMA) [11-15], on the other hand, implements an inverse reliability analysis, which determines the performance measure of a target design point. Probabilistic constraints are then formulated from these performance measures. Observations of the strengths and weaknesses of both RIA and PMA has led to the development of Hybrid Reliability Approach (HRA) [16, 17]. HRA uses a selection factor to determine whether PMA or MRIA would be efficient to use. Derivations of a Unified Reliability Formulation (URF) [18] had revealed how various RBDO algorithms can be reformulated into one general equation based on the reliability analyses in the standard normal design space and the fundamental aspect of linear expansions with allowable reliabilities.
Recently, the design optimization problems with arbitrarily distributed uncertainties have drawn high attentions [19-21] because they cannot be properly solved by RBDO algorithms that require transformation of the original design space to the standard normal design space. Therefore, the situations such as unknown random distribution type, undeterminable transformation to the standard normal design space, and insufficient information about the random distribution, are difficult for most RBDO algorithms. Thus, a new method that is capable of efficiently solving the RBDO problem with arbitrarily distributed uncertainties in the original design space is desirable.
In this paper, a novel method called Ensemble of Gradient-based Transformed Reliability Analyses (EGTRA) is derived based on estimation of arbitrary distribution in the original design space using Kernel Density Estimation

(KDE) [22, 23] and reliability analyses along the gradient-based transformed direction [24]. A linearly approximated probabilistic constraint is formulated at the design point determined at the allowable reliability level. The aforementioned derivations and technologies are conducted in the original space; therefore, EGTRA is expected to be capable of solving RBDO problems with arbitrarily distributed uncertainties. Two numerical examples are solved in order to investigate the numerical performances of the proposed method.

4. Derivation of Ensemble of Gradient-based Transformed Reliability Analyses (EGTRA)

A typical RBDO problem is formulated as follows:

$$Min \quad f(\mathbf{d}) \quad s.t. \quad P\left[g_i(\mathbf{X})>0\right] \le P_{f,i} \qquad i=1...M \tag{1}$$

where $\mathbf{X}$ is the randomly distributed design variable; $\mathbf{d}$ is the mean of $\mathbf{X}$ and is commonly used as the design variable; f is the objective (or cost) function; g_i is the i^{th} constraint function and $g_i \le 0$ represents the i^{th} safe region; $P_{f,i}$ is i^{th} allowable failure probability. There are M constraints and L variables. Several RBDO algorithms have been investigate the failure probability in Eq. (1) during the optimization process to reach the optimality, feasibility and reliability simultaneously. However, some existing methods cannot perform the reliability analysis properly when the following conditions occur: (1) Distribution type of $\mathbf{X}$ is unknown. (2) Transformation of $\mathbf{X}$ to the i^{th} standard normal design space $\mathbf{U}_i$ is difficult to determine or does not exist. (3) Data about the random distribution is insufficient.

This paper assumes the relative positions between the sampling points and the design point remain constant and are independent from the location of the design point. The following gradient-based transformation [24] is first considered to transform the original coordinate to the single variate design space y_i toward the direction of the i^{th} constraint gradient $\mathbf{v}_i = \nabla_{\mathbf{x}} g_i(\mathbf{d}) \left\| \nabla_{\mathbf{x}} g_i(\mathbf{d}) \right\|^{-1}$:

$$y_{i,p} = (\mathbf{s}_p - \mathbf{d}) \cdot \mathbf{v}_i \tag{2}$$

where the design point $\mathbf{d}$ is the origin of the i^{th} gradient-based transformed design space, denoted by Ω_i; the value of $y_{i,p}$ is the projection of the p^{th} sampling point in Ω_i. Figure 1 (a) illustrates the transformation to the constraint gradient direction and the mapping of the sampling points to Ω_i.

A Most Probable Target Point (MPTP) in Ω_i, denoted by $y_i^{\#}$, is defined such that the summation of the cumulative probability of each kernel function from the MPTP to infinity is equal to the allowable failure probability, as illustrated in Figure 1 (b). Therefore, $y_i^{\#}$ is determined by solving the following equation:

$$N^{-1} \sum_{p=1}^{N} \Phi_p(y_i^{\#}) = P_{f_i} \tag{3}$$

In Eq. (3), Φ_p is the p^{th} Gaussian Cumulative Distribution Function (CDF) and is defined as below:

$$\Phi_p(y_0) = \int_{y_0}^{\infty} \sigma^{-1} (\sqrt{2\pi})^{-1} \exp[-0.5 (y_i - y_{i,p})^2 \sigma^{-2}] dy_i \tag{4}$$

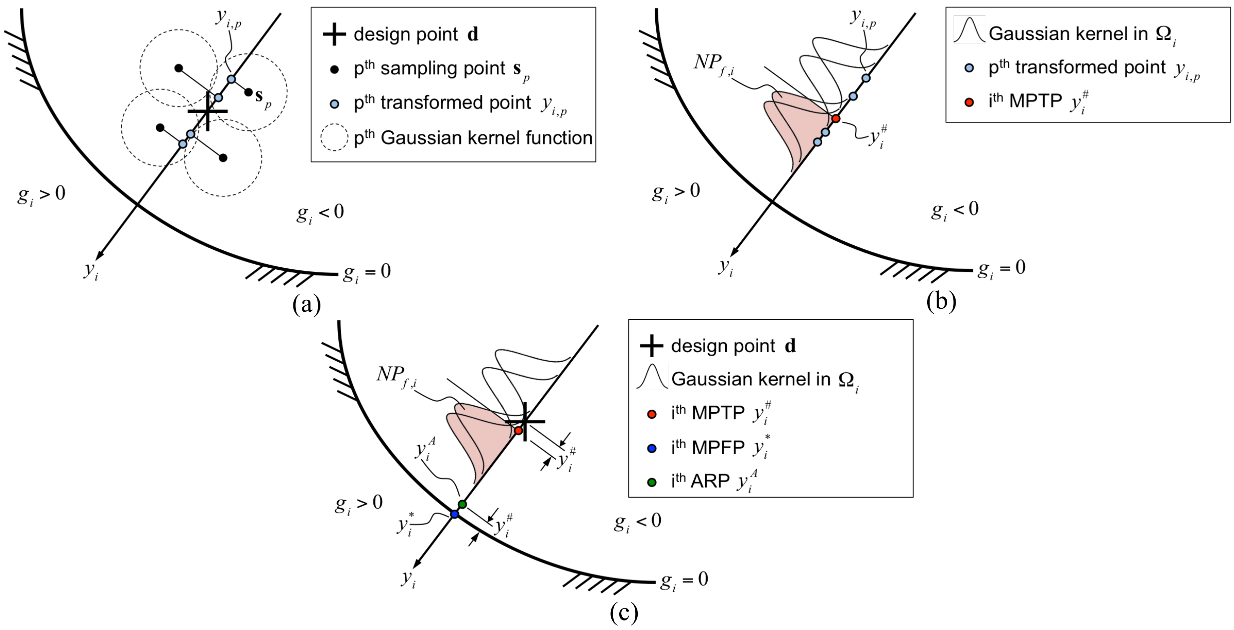

Figure 1. Illustration of the proposed method in Ω_i: (a) Transformation of sampling points to Ω_i; (b) Determination of Most Probable Target Point; (c) Determination of Allowable Reliability Point.

where σ is a shape parameter in KDE [22, 23]. The size of σ is critical for the accuracy of the estimation of the arbitrarily distributed PDF. The location of MPTP is essential for the estimation of the design point with allowable reliability and the further formulation of probabilistic constraint.

The RBDO process in Ω_i is expected to move the MPTP to the limit state; therefore, the value of $y_i^{\#}$ also represents the allowable tolerance near the limit state, as illustrated in Figure 1 (c). The i^{th} Allowable Reliability Point (ARP) y_i^A is then determined by

$$y_i^A = y_i^* - y_i^{\#} = -g_i(\mathbf{d}) \| \nabla_{\mathbf{x}} g_i(\mathbf{d}) \|^{-1} - y_i^{\#} \tag{5}$$

where y_i^* is the Most Probable Failure Point (MPFP) in Ω_i. In the end, the RBDO procedure is expected to move the design point to the location of the ARP. Thus, the failure probability of the new design point is expected to reach the allowable level.

Using URF [18], the linearly approximated probabilistic constraint is formulated below:

$$(\mathbf{d} - \mathbf{x}_i^A) \cdot \mathbf{v}_i \le 0 \tag{6}$$

where $\mathbf{x}_i^A$ is the i^{th} Allowable Reliability Point (ARP) in the original design space. Therefore, a final formulation of the i^{th} probabilistic constraint using the EGTRA is given as follows:

$$\{\mathbf{d} - \mathbf{d}^{(k)} + [g_i(\mathbf{d}^{(k)}) \| \nabla_{\mathbf{x}} g_i(\mathbf{d}^{(k)}) \|^{-1} + y_i^{\#}]\mathbf{v}_i^{(k)}\} \cdot \mathbf{v}_i^{(k)} \le 0 \tag{7}$$

From the derived Eq. (7), it is noted that $y_i^{\#}$ is not only a most probable allowable tolerance but also a newly defined reliability assessment in the original design space for the proposed EGTRA method.

5. Numerical Examples

This section first introduces the procedure of random distribution generation that is used in this paper. Two mathematical problems are then solved by the proposed EGTRA.

5.1 Generations of Arbitrarily Distributed Random Variables

This paper considers the design point at the mean value of the generated random distribution. All problems are solved in the two-dimensional design space, i.e. $L = 2$, for better illustration of the results. The following distributions are artificially generated by specified procedures for research purpose and may not be seen in real world. This paper intentionally generates some distributions that are "concavely" ranged and are difficult for conventional reliability analyses.

Figure 2 (a) shows the generated heat-shaped distribution with its mean point at the origin of the coordinate. If it is improperly considered as a Gaussian distribution, as shown by the dashed contour, the standard deviations along the x and y directions will be computed as [0.3896, 0.1648], respectively. Figure 2 (b) to (d) show the generated "like"-shaped, star-shaped and corona-shaped distributions, respectively. If these distributions are improperly considered as Gaussian distributions, the standard deviations along the x and y directions are computed as [0.2493, 0.3045], [0.3596, 0.3606] and [0.4329, 0.4337] for the "like"-shaped, star-shaped and corona-shaped distributions, respectively.

The heart-shaped distribution shows a concave distribution supported in the entire design space. The "like"-shaped distribution shows a combined distribution that is partially supported in in the entire domain and partially supported in a semi-interval. The star-shaped distribution shows a uniform distribution in a concave region. The corona-shaped distribution shows a distribution supported in the entire design space with a void region at the center.

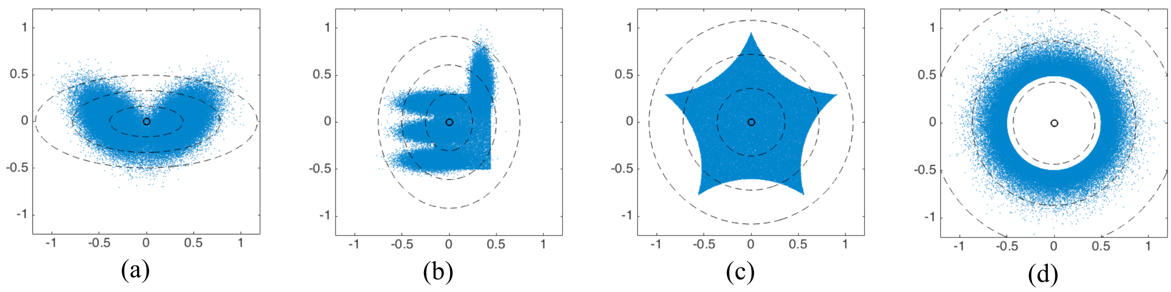

Figure 2. Generated (a) heart-shaped, (b) "like"-shaped, (c) star-shaped and (d) corona-shaped distributions.

5.2 Example 1: A Linear Math Problem

The first example is a linear mathematical problem [10, 11], which is shown in Eq. (8).

$$\begin{aligned} &\underset{d}{Min} \quad d_1 + d_2 \\ &s.t. \quad P[g_1 = -X_1 - 2X_2 + 10 > 0] \le P_f; \quad P[g_2 = -2X_1 - X_2 + 10 > 0] \le P_f; \quad 0.1 \le d_1, d_2 \le 10 \end{aligned} \tag{8}$$

In this paper, two various levels of allowable failure probabilities are investigated: $P_f = 1\%$ and 30%. The initial

design point is located at [5, 5]. Figure 3 shows the optimal solutions using EGTRA with $N = 50000$ to solve the problem in Eq. (8) in the generated random distributions. The subfigures (a) to (d) are the results for $P_f = 1\%$ while the subfigures (e) to (h) are for $P_f = 30\%$. The red lines represent the limit states of the original constraints and the black lines are the linearly approximated probabilistic constraints, which are determined using the Eq. (7)
.

Because each kernel reliability analysis in EGTRA is completed along the same gradient direction, the required function evaluation of each constraint per iteration is only 3. Monte Carlo Simulations (MCS) with 10^5 sampling points were used to evaluate the true failure probabilities. The numerical results showed the proposed method is capable of solving the problems with the generated design uncertainties with accurate estimations of reliabilities in the original design space, i.e. errors were less than 1%.

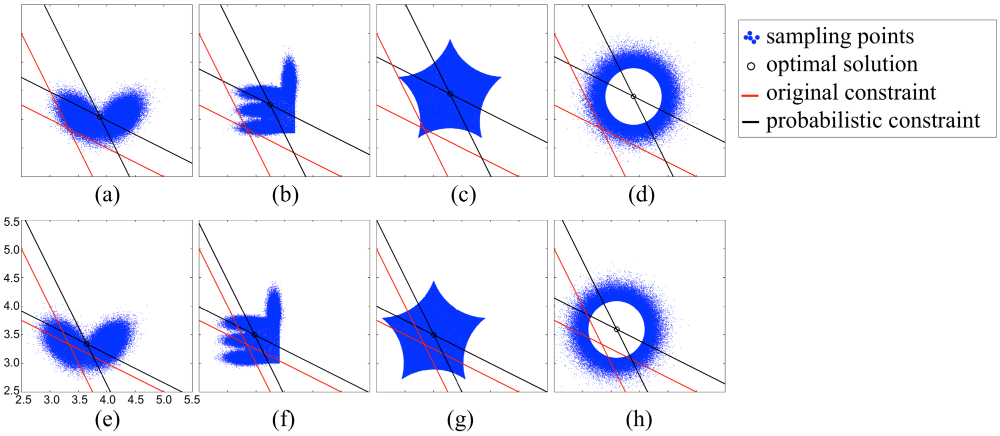

Figure 3. Solutions of example 1 in various distributions and allowable failure probabilities: (a) heart, 1%; (b) “like”, 1%; (c) star, 1%; (d) corona, 1%; (e) heart, 30%; (f) “like”, 30%; (g) star, 30%; (h) corona, 30%.

5.3 Example 2: A Nonlinear Benchmark Problem

The second example is a well-known benchmark mathematical problem [10, 11, 25, 26] that contains three nonlinear constraints. Because the third constraint is inactive, it is removed and the following problem formulation is considered in this paper.

$$\begin{aligned} &\underset{d}{Min} \quad d_1 + d_2 \\ &s.t. \quad P[g_1 = 1 - (X_1^2 X_2)/20 > 0] \le P_f; \quad P[g_2 = 1 - (X_1 + X_2 - 5)^2/30 - (X_1 - X_2 - 12)^2/120 > 0] \le P_f; \quad 0.1 \le d_1, d_2 \le 10 \end{aligned} \tag{9}$$

In this problem, two various levels of allowable failure probabilities, i.e. $P_f = 1\%$ and 30%, are studied. EGTRA is used to solve the problem with $N = 50000$. The initial design point is located at [5, 5]. The rest of the problem setting is the same as the first example. Figure 4 (a) to (d) show the optimal solutions for $P_f = 1\%$ while the subfigures (e) to (h) show the ones for $P_f = 30\%$. EGTRA was capable of solving the given problems with only $3M$ function evaluations per iteration. However, the accuracy of EGTRA slightly dropped because linear approximations were utilized for the nonlinear constraints in Eq. (9), i.e. error increased up to around 6%.

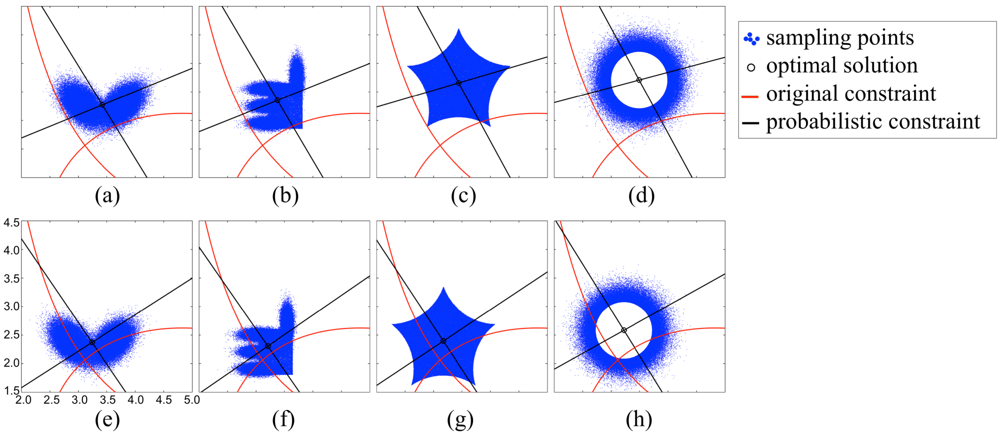

Figure 4. Solutions of example 2 in various distributions and allowable failure probabilities: (a) heart, 1%; (b) “like”, 1%; (c) star, 1%; (d) corona, 1%; (e) heart, 30%; (f) “like”, 30%; (g) star, 30%; (h) corona, 30%.

6. Conclusions

Some existing RBDO algorithms transformed the original random design space to the standard normal design space in order to perform the reliability analyses for the evaluation of failure probabilities. However, these reliability analyses cannot be properly executed when the transformation to the standard normal design space cannot be determined. A new RBDO algorithm, Ensemble of Gradient-based Transformed Reliability Analyses (EGTRA), was developed to solve design optimization problems with arbitrarily distributed uncertainties in the original design space. The arbitrarily distributed PDF was approximated by KDE and then transformed to a single-variate coordinate toward the constraint gradient direction. The entire reliability analysis is decomposed to multiple kernel reliability analyses in the gradient-based transformed design space and the results are merged together for the formulation of a linearly approximated probabilistic constraint function. Because each kernel reliability analysis is completed along the same gradient direction, the required function evaluation of each constraint per iteration is only 3. The numerical results showed the proposed method is capable of solving the problems with the generated design uncertainties with accurate estimations of reliabilities in the original design space.

The newly developed method does not need transformation to the standard normal design space and reliability analyses that require additional function evaluations. EGTRA is able to perform very accurate reliability analysis for linear RBDO problems and the accuracy reduces when the constraints are nonlinear. EGTRA is a method that requires information about the sampling points of the arbitrarily distributed random variables. Insufficient sampling points may lead to inaccurate estimations of PDF and failure probabilities. The performance of EGTRA will reduce when the sampling at the tail of the distribution is insufficient. This may happen when the amount of sampling points and the level of allowable failure probability are both very low.

7. Acknowledgments

The supports from Ministry of Science and Technology, Taiwan (grant number MOST 103-2221-E-033-015), Research and Development Center for Microsystem Reliability, and Center for Robotics Research at Chung Yuan Christian University, Taiwan are greatly appreciated.

8. References

[1] H.-S. Jung and S. Cho, "Reliability-based topology optimization of geometrically nonlinear structures with loading and material uncertainties," *Finite elements in analysis and design*, vol. 41, pp. 311-331, 2004.

[2] B. Ellingwood, *Development of a probability based load criterion for American National Standard A58: Building code requirements for minimum design loads in buildings and other structures*, 577, US Department of Commerce, National Bureau of Standards, 1980.

[3] R. D. Pope and R. A. Kramer, "Production uncertainty and factor demands for the competitive firm," *Southern Economic Journal*, pp. 489-501, 1979.

[4] S. S. Hasan, J. M. Leith, B. Campbell, K. L. Smith, and F. A. Matsen III, "Characteristics of unsatisfactory shoulder arthroplasties," *Journal of shoulder and elbow surgery*, vol. 11, pp. 431-441, 2002.

[5] M. Karamouz and H. V. Vasiliadis, "Bayesian stochastic optimization of reservoir operation using uncertain forecasts," *Water Resources Research*, vol. 28, pp. 1221-1232, 1992.

[6] E. Nikolaidis and R. Burdisso, "Reliability Based Optimization: A Safety Index Approach," *Computers & Structures*, vol. 28, pp. 781-788, 1988.

[7] I. Enevoldsen, "Reliability-Based Optimization as an Information Tool," *Mechanics of Structures and Machines*, vol. 22, pp. 117-135, 1994.

[8] I. Enevoldsen and J. D. Sorensen, "Reliability-Based Optimization in Structural Engineering," *Structural Safety*, vol. 15, pp. 169-196, 1994.

[9] S. V. L. Chandu and R. V. Grandhi, "General Purpose Procedure for Reliability Based Structural Optimization under Parametric Uncertainties," *Advances in Engineering Software*, vol. 23, pp. 7-14, 1995.

[10] P. T. Lin, H. C. Gea, and Y. Jaluria, "A Modified Reliability Index Approach for Reliability-Based Design Optimization," *Journal of Mechanical Design*, vol. 133, 044501, 2011.

[11] J. Tu, K. K. Choi, and Y. H. Park, "A New Study on Reliability Based Design Optimization," *Journal of Mechanical Design*, vol. 121, pp. 557-564, 1999.

[12] B. D. Youn and K. K. Choi, "An Investigation of Nonlinearity of Reliability-Based Design Optimization Approaches," *Journal of Mechanical Design*, vol. 126, pp. 403-411, 2004.

[13] B. D. Youn and K. K. Choi, "Selecting Probabilistic Approaches for Reliability-Based Design Optimization," *AIAA Journal*, vol. 42, pp. 124-131, 2004.

[14] B. D. Youn, K. K. Choi, and L. Du, "Adaptive Probability Analysis Using an Enhanced Hybrid Mean Value Method," *Structural and Multidisciplinary Optimization*, vol. 29, pp. 134-148, 2005.

[15] B. D. Youn, K. K. Choi, and L. Du, "Enriched Performance Measure Approach for Reliability-Based Design Optimization," *AIAA Journal*, vol. 43, pp. 874-884, 2005.

[16] P. T. Lin, Y. Jaluria, and H. C. Gea, "A Hybrid Reliability Approach for Reliability-Based Design

Optimization," in *ASME 2010 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference, IDETC/CIE 2010*, Montreal, Quebec, Canada, DETC2010-28871, 2010.
[17] P. T. Lin, M. C. E. Manuel, Y.-H. Liu, Y.-C. Chou, Y. Ting, S.-S. Shyu, C.-K. Chen, and C.-L. Lee, "A Multifaceted Approach for Safety and Reliability-Based Design Optimization," *Mathematical Problems in Engineering*, 817820, 2014.
[18] P. T. Lin, "Ensemble of Unified Reliability Formulations (EURF)," in *10th World Congress on Structural and Multidisciplinary Optimization, WCSMO10*, Orlando, FL, USA, 5476, 2013.
[19] X. Lai, "A modified method to compute the reliability with arbitrary independent random variables," *Proceedings of the Institution of Mechanical Engineers, Part E: Journal of Process Mechanical Engineering*, vol. 228, pp. 320-331, 2014.
[20] P. T. Lin and S.-P. Lin, "Ensemble of Gaussian-based Reliability Analyses (EoGRA) in the Original Design Space for Design Optimization Under Arbitrarily Distributed Design Uncertainties," in *The 8th China-Japan-Korea Joint Symposium on Optimization of Structural and Mechanical Systems, CJK-OSM8*, Gyeongju, Korea, 2014.
[21] P. T. Lin and S.-P. Lin, "Reliability-Based Design Optimization in X-Space Using Ensemble of Gaussian Reliability Analyses (EoGRA)," in *ASME 2015 International Design and Engineering Technical Conferences & Computers and Information in Engineering Conference, IDETC/CIE 2015*, Boston, MA, USA, DETC2015-46130, 2015.
[22] M. Rosenblatt, "Remarks on some nonparametric estimates of a density function," *The Annals of Mathematical Statistics*, vol. 27, pp. 832-837, 1956.
[23] E. Parzen, "On estimation of a probability density function and mode," *The annals of mathematical statistics*, pp. 1065-1076, 1962.
[24] P. T. Lin and H. C. Gea, "A Gradient-Based Transformation Method in Multidisciplinary Design Optimization," *Structural and Multidisciplinary Optimization*, vol. 47, pp. 715-733, 2013.
[25] R. J. Yang and L. Gu, "Experience with Approximate Reliability-Based Optimization Methods," *Structural and Multidisciplinary Optimization*, vol. 26, pp. 152-159, 2004.
[26] H. C. Gea and K. Oza, "Two-Level Approximation Method for Reliability-Based Design Optimisation," *International Journal of Materials and Product Technology*, vol. 25, pp. 99-111, 2006.