Adjoint Methods of Sensitivity Analysis for Lyapunov Equation

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1. Abstract
The existing direct sensitivity analysis of optimal structural vibration control based on Lyapunov's second method is computationally expensive when applied to finite element models with a large number of degree-of-freedom and design variables. A new adjoint sensitivity analysis method is proposed in this paper. Using the new method the sensitivity of the performance index, a time integral of a quadratic function of state variables, with respect to all design variables is calculated by solving two Lyapunov matrix equations. Two numerical examples demonstrate the accuracy and efficiency of the proposed method. Finally, we use the adjoint sensitivity analysis scheme to solve a topology optimization problem.
2. Keywords: Adjoint method, sensitivity analysis, topology optimization

3. Introduction
In time domain, there is a classic problem formulation of passive structural vibration control that deals with the dynamic system disturbed by initial conditions. The objective is to find design parameters of the damped vibration system that minimize the performance index in the form of time integral of the quadratic function of state variables (displacement and velocities, e.g. see Eq.(5)). This performance index can be evaluated by Lyapunov's second method [1].
Based on the Lyapunov equation, the evaluation of performance indices are simplified into matrix quadratic forms and do not require the time domain integration. Parameter optimization problems with a quadratic performance index have been solved by this method [2~8]. Wang et al. [9] applied the Lyapunov equation to solve the transient response optimization problem of linear vibrating systems excited by initial conditions. In their work, the Lyapunov equation was expanded to a set of linear equation and direct sensitivity was carried out by use of the same system of linear equation. The computational effectiveness of the method is illustrated by applying it to the classical vibration absorber and to a cantilever beam carrying an absorber at its midpoint. Du [10] applied the Lyapunov equation to obtain the optimum configuration of dynamic vibration absorber (i.e., DVA) attached to an undamped or damped primary structure. The Lyapunov equation is also used in other fields of optimal design.
In this paper, we consider one case of passive control optimization problem, that is, to minimize an integrated quadratic performance measure for damped vibrating structures subjected to initial conditions. The goal of this paper is to present an adjoint sensitivity analysis method considering the above mentioned objective function based on Lyapunov's second function. The results indicate the potential of application of the proposed method to topology optimization under the special time domain criterion.

4. Application of Lyapunov's second method to optimize transient response of mechanical systems
Consider a viscously damped linear vibration system governed by the equation:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0} \tag{1}$$

where $\mathbf{M}(N\times N)$ is the mass matrix, $\mathbf{C}(N\times N)$ is the damping matrix, $\mathbf{K}(N\times N)$ is the stiffness matrix, and $\mathbf{u}(N\times 1)$ is displacement vector. N is the structural degree of freedoms.
Assume the system is excited by initial displacements or velocities. The design problem is to find in **M**, **K** and **C** matrices to minimize a performance matrix in the form

$$J = \int_0^T q(\mathbf{u}, \dot{\mathbf{u}})\,\mathrm{d}t \tag{2}$$

where, $q(u, \dot{u}) = \mathbf{u}^{\mathrm{T}}\mathbf{Q}_u\mathbf{u} + \dot{\mathbf{u}}^{\mathrm{T}}\mathbf{Q}_{\dot{u}}\dot{\mathbf{u}}$ is a quadratic function of $\mathbf{u}$ and $\dot{\mathbf{u}}$. Transient dynamic responses have to be performed to evaluate the objective function. Direct or adjoint methods can be applied to evaluate the response sensitivity required for evaluation sensitivity of the performance. Alternative, if we replace the upper bound of integration to infinite, we can use Lyapunov's second method to evaluate the performance without performing transient dynamic response analysis.
To apply Lyapunov's second method to this system, it is necessary to rewrite Eq.(1) in the state space form

$$\dot{\mathbf{X}} = \mathbf{AX} \tag{3}$$

Where

$$\mathbf{A} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix} \tag{4}$$

The matrix **A** is ($2N\times2N$). The vector **X** is ($2N\times1$). Structural design parameters such as mass density, damping ratio and spring stiffness are contained in the matrix **A**. The optimization problem is to choose these parameters to minimize the performance measure ***J*** defined by

$$J = \int_0^\infty \mathbf{X}^{\mathrm{T}}\mathbf{QX}\,\mathrm{d}t \tag{5}$$

for a given initial excitation **X**(0). In Eq.(5), $\mathbf{Q}(2N\times2N)$ is a positive semi-definite symmetric weighting matrix and t denotes time. According to Lyapunov theory of stability, for an asymptotically stable system, there exist a symmetric positive semi-definite matrix $\mathbf{P}(2N\times2N)$ satisfying

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{PA} = -\mathbf{Q} \tag{6}$$

Eq.(6) is the well-known Lyapunov equation. Based on the Lyapunov's second equation, the Eq.(5) can be further simplified as

$$J = \mathbf{X}(0)^{\mathrm{T}}\mathbf{PX}(0) \tag{7}$$

That is to say, to minimize ***J*** in Eq.(5) is equivalent to minimize $\mathbf{X}(0)^{\mathrm{T}}\mathbf{PX}(0)$, where $\mathbf{X}(0)$ is the initial state vector and the unknown symmetric matrix **P** can be obtained by solving Eq.(6).

5. Sensitivity analysis scheme

To apply gradient-based optimization method to solve the above optimization problem, sensitivity (gradient) of the objective functions with respect to the design variables is needed. The adjoint method will be developed in this paper. The new method just needs to solve the Lyapunov function twice to obtain the sensitivities with respect to all the design variables.

For ease of presentation of the new sensitivity analysis scheme, we adopt Du's approach of using Kronecker product and column expansion to expand the Lyapunov equation as a system of linear equation. The column expansion of matrix **V** is defined as a vector that stacks all columns of this **V** matrix. For example, for the 3×3 matrix **V**

$$\mathbf{V} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix} \tag{8}$$

the column expansion $cs(\mathbf{V})$ of **V** is

$$\overline{\mathbf{V}} = cs(\mathbf{V}) = \begin{bmatrix} V_{11} & V_{12} & V_{13} & V_{21} & V_{22} & V_{23} & V_{31} & V_{32} & V_{33} \end{bmatrix}^{\mathrm{T}} \tag{9}$$

Note that $cs(\mathbf{V})$ is a 9×1 vector. The operator $cs(*)$ refers to the expansion operation. For an N-dof system, using the Kronecker product, (6) can be written as

$$\mathbf{G}\overline{\mathbf{P}} = -\overline{\mathbf{Q}} \tag{10}$$

where $\overline{\mathbf{P}} = cs(\mathbf{P})$, $\overline{\mathbf{Q}} = cs(\mathbf{Q})$ and the matrix **G** can be obtained from the matrix **A** by Kronecker product. That is

$$\mathbf{G} = \left(\mathbf{A}^{\mathrm{T}} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{A}^{\mathrm{T}}\right) \tag{11}$$

and $\mathbf{E}(2N\times2N)$ is an identity matrix with the same size of **A**. Now, by direct calculation, the objective function in Eq.(7) can be written as

$$J = \overline{\mathbf{S}}^{\mathrm{T}}\overline{\mathbf{P}} \tag{12}$$

where

$$\overline{\mathbf{S}} = cs(\mathbf{S}), \quad \mathbf{S} = \mathbf{X}(0)\mathbf{X}(0)^{\mathrm{T}} \tag{13}$$

S also is a positive semi-definite symmetric matrix as matrix **Q**. From the Eq.(10), the term $\frac{\partial \overline{\mathbf{P}}}{\partial d_k}$ can be obtained by

$$\frac{\partial \overline{\mathbf{P}}}{\partial d_k} = -\mathbf{G}^{-1}\left(\frac{\partial \mathbf{G}}{\partial d_k}\overline{\mathbf{P}} + \frac{\partial \overline{\mathbf{Q}}}{\partial d_k}\right) \tag{14}$$

Thus sensitivity can be expressed as

$$\frac{\partial J}{\partial d_k} = \mathbf{X}(0)^{\mathrm{T}}\frac{\partial \mathbf{P}}{\partial d_k}\mathbf{X}(0) = -\overline{\mathbf{S}}^{\mathrm{T}}\mathbf{G}^{-1}\left(\frac{\partial \mathbf{G}}{\partial d_k}\overline{\mathbf{P}} + \frac{\partial \overline{\mathbf{Q}}}{\partial d_k}\right) \tag{15}$$

The right hand of Eq.(15) can be rewritten as

$$\frac{\partial J}{\partial d_k} = \mathbf{X}(0)^{\mathrm{T}}\frac{\partial \mathbf{P}}{\partial d_k}\mathbf{X}(0) = \overline{\boldsymbol{\lambda}}^{\mathrm{T}}\overline{\mathbf{D}}^k \tag{16}$$

where

$$\overline{\boldsymbol{\lambda}}^{\mathrm{T}} = -\overline{\mathbf{S}}^{\mathrm{T}}\mathbf{G}^{-1} \tag{17}$$

$$\overline{\mathbf{D}}^k = \left(\frac{\partial \overline{\mathbf{Q}}}{\partial d_k} + \frac{\partial \mathbf{G}}{\partial d_k}\overline{\mathbf{P}}\right) \tag{18}$$

Note that $\overline{\boldsymbol{\lambda}}$ and $\overline{\mathbf{D}}^k$ are the column expansion of matrices $\boldsymbol{\lambda}$ and $\mathbf{D}^k$, respectively.

$$\mathbf{A}\boldsymbol{\lambda} + \boldsymbol{\lambda}\mathbf{A}^{\mathrm{T}} + \mathbf{S} = 0 \tag{19}$$

$\boldsymbol{\lambda}$, the adjoint matrix, can be obtained by solving the above Lyapunov matrix equation. $\mathbf{D}^k$ can be also computed by

$$\mathbf{D}^k = \frac{\partial \mathbf{Q}}{\partial d_k} + \frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial d_k}\mathbf{P} + \mathbf{P}\frac{\partial \mathbf{A}}{\partial d_k} \tag{20}$$

Finally, the sensitivity of the objective function with respect to the design variable can be expressed as

$$\frac{\partial J}{\partial d_k} = \frac{\partial \mathbf{X}(0)^{\mathrm{T}}}{\partial d_k}\mathbf{P}\mathbf{X}(0) + \mathbf{X}(0)^{\mathrm{T}}\mathbf{P}\frac{\partial \mathbf{X}(0)}{\partial d_k} + \sum_{i=1}^{2N}\sum_{j=1}^{2N}\lambda_{ij}D_{ij}^k \tag{21}$$

For the case $\mathbf{X}(0)$ independent of design variables, the Eq.(21) can simplified as

$$\frac{\partial J}{\partial d_k} = \sum_{i=1}^{2N}\sum_{j=1}^{2N}\lambda_{ij}D_{ij} \tag{22}$$

6. Numerical example

Two examples are presented in this section. The first example is used to demonstrate the accuracy and efficiency of the proposed methods. The optimal support location is solved as a topology optimization problem in the second example.

6.1. Example 1

In this example, we consider a clamped-free beam (3m×0.02m×0.02m) attached with several identical damped springs (along Y direction). The beam material is linear elastic with the elastic modulus 2.1×10^{11}Pa and mass density 7850Kg/m^3. The spring stiffness k_s is to be determined (N/m), and the damping coefficient is 10^3N·s/m. Figure 1 shows the beam model used in this example. Specially, the beam is uniformly meshed into 50 2-node beam elements. Each node has 2 DOFs (lateral displacement and rotation about Z-axis). Five equally spaced damped spring supports are considered. The initial displacements and velocities of all nodes are zero and 10m/s respectively. The stiffness k of each spring is chosen as the design variable. Thus, there are 5 design variables.
Firstly, we compare the sensitivity results from three methods, central difference method, adjoint method and direct method to validate the proposed adjoint method.

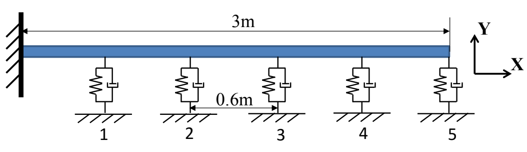

Figure 1: The beam model with 5 damped springs

The objective function is

$$J = \sum_{i=1}^{R} \int_0^{\infty} y_i^2 dt \tag{23}$$

where y_i is the Y-direction displacement of the ith node of the beam, R is the total number of the free nodes of the beam.

To study the effect of step size in central difference analysis, we calculate the approximation of sensitivity of spring stiffness of the spring at right hand of beam at k_s=10^5N/m for three different step sizes. The results are shown in table 1. We chooseΔ k_s=100N/m for further study in this example.

Table 1: Sensitivity results at k_s=1.0×10^5N/m from different step sizes

Step size	10^4	10^3	10^2
Sensitivity result	-2.4858×10^{-11}	-2.4715×10^{-11}	-2.4714×10^{-11}

The sensitivity results of the objective function with respect to k_s of each spring at k_s=1.0×10^5N/m from central difference method, adjoint method and direct method are shown in figure 2 and are represented by the black crosses, red squares and blue rounds, respectively. The results show that the adjoint method obtains identical results with the central difference method and direct method.

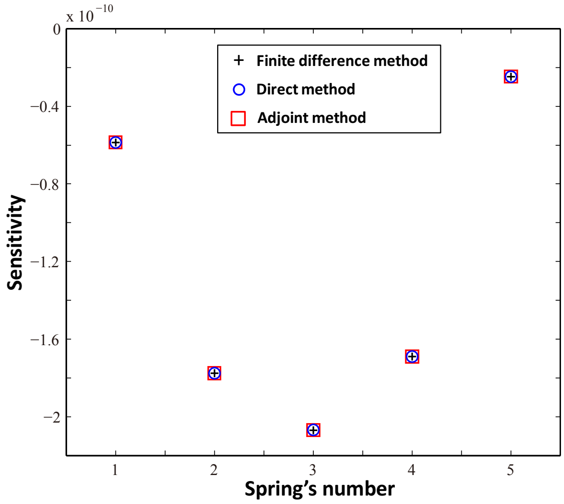

Figure 2: Sensitivity results of the stiffness k of each spring from three methods

In this paper, the CPU time results are the average values of CPU time of 10 repeated analyses. The computer used in this paper is i7-3770 3.4GHz, Windows 7.

Now, we compare the CPU time of direct method and adjoint method. The CPU time results of these two methods are summarized in table 2. The results show that total CPU time of the sensitivity analysis process of adjoint method is less than direct method, especially when the problem has large number of DOFs. T_A is the CPU time of the sensitivity analysis process of adjoint method, and T_D is the CPU time of the sensitivity analysis process of direct method.

Table 2: CPU time of two methods vs. number of DOFs in the model

Number of DOFs	CPU time (s)		T_A/T_D
	T_A(adjoint)	T_D(direct)	
100	0.091	0.232	39.34%
600	22.095	61.802	35.75%

6.2. Example 2

Topology optimization problems always have large numbers of design parameters. We construct a topology optimization problem to test the new sensitivity analysis methods. In this example, we consider a 1m$\times$1m$\times$0.01m plate attached with several identical damped springs(k_s=10^6N/m, c_s=100N·s/m). One edge of the plate is clamped and other three edges are free. The material of the plate is linear elastic with elastic modulus 2.1×10^{11}Pa, Poisson ratio 0.3, and mass density 7850Kg/m^3. The initial velocity of Z direction of all the free nodes of plate is 10m/s. The design problem is to decide the optimized location of *H* damped springs to minimize a criterion defined below. We formulate this problem as a topology optimization problem. This is achieved by introducing an artificial density variable to describe the spatial distribution of the damped springs and use interpolation model of SIMP to obtain 0-1 design. Specifically, an identical potential damped spring (along Z direction) is placed between every free nodes and the ground.

Set a virtual density ρ_i to every spring as the design variable. ρ_i is a continuous variable, and $\rho_i \in [\rho_{\min}, 1]$. We introduce an artificial relation between density (ρ_i) and the parameters of the damped springs.

$$K_i = \rho_i^l K_0,\ C_i = \rho_i^l C_0 \tag{24}$$

where *l* is the penalty parameter. In this example, *l* is chosen as 1.2. The analysis model is shown in Figure 3, where the purple lines are the dumped springs and the blue square elements are the 4-node square plate elements (shell63 in Ansys). Each node of the element has 3 DOFs, u_z , θ_x and θ_y . The element size of the plate is 0.1m (there are 110 free nodes). The analysis model has 330 DOFs and 110 design parameters. The topology optimization problem can be expressed as

$$\begin{aligned} \min \quad & J = \sum_{i=1}^{110} \int_0^\infty z_i^2 dt \\ const. \quad & \sum_{i=1}^{M} \rho_i = H \\ & 0 < \rho_{\min} \le \rho_i \le 1 \end{aligned} \tag{25}$$

where *H* specifies the material volume available for the damped springs. Here we assume each spring, if any, uses material volume 1, *H* will be the number of damped springs in the final optimum design. z_i is the Z-direction displacement of the *i*th node of the plate. The model has 110 free nodes, so the objective function concerns the displacements of all the free nodes. It should be noted this example mainly serves to compare different methods of sensitivity analysis through solving the topology optimization problem described by Eq.(25).

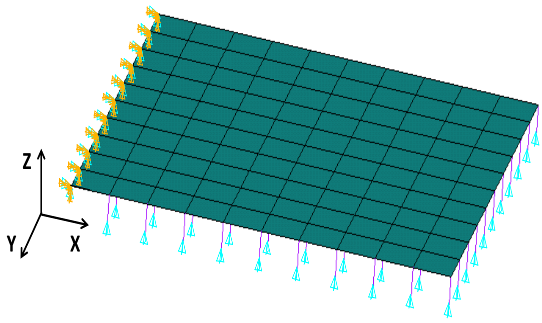

Figure 3: The analysis model in topology optimization

The CPU time of solving processes of adjoint method and direct method is summarized in table 3. The computing time of adjoint method is much shorter than the computing time of direct method.

Table 3: CPU time of sensitivity analysis of two methods

	Adjoint	Direct
CPU time (s)	10.575	181.130

Finally, we use above mentioned four sensitivity analysis schemes to solve the topology optimization described in Eq.(25). *H* is set to 2. Figure 4 shows that optimization using different sensitivity analysis methods have identical iteration histories and obtain same optimized designs. The CPU time of solving processes of topology optimization problem using different sensitivity analysis methods is summarized in table 4. The optimization process using adjoint method just takes about 10 minutes which is far less than the CPU time of other one.

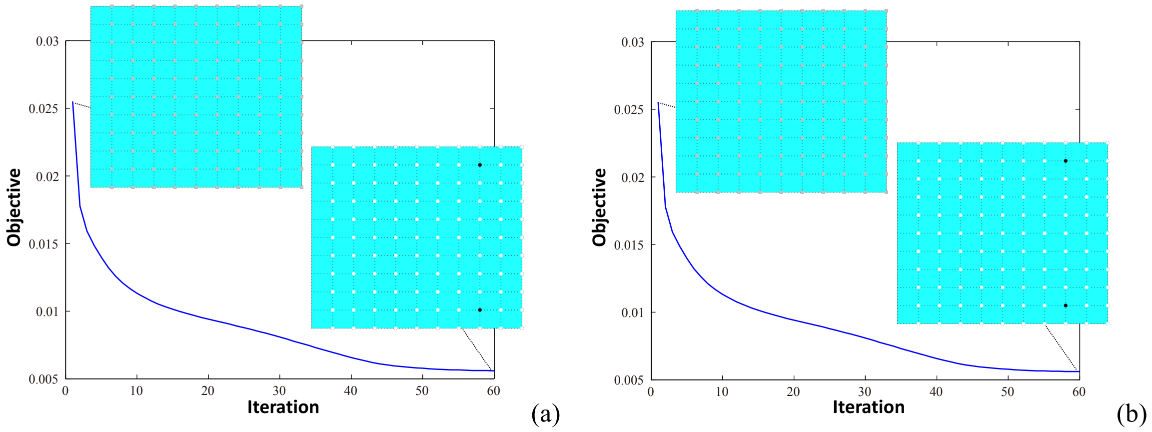

Figure 4: Iteration histories of the objective function of optimization process: (a) Ajoint method; (b) Direct method

Table 4: CPU time of optimization processes using different sensitivity analysis methods

	AVMF	DVMF
CPU time (s)	636.852	10886.824

7. Conclusion

A new adjoint sensitivity analysis method for the integral square performance index is proposed in this paper. The new approach requires the solutions of two Lyapunov equations only, one for the performance index and one for the adjoint vector. In contrast, direct sensitivity analysis requires the solution of a Lyapunov equation for each design variable. This improvement in computational efficiency
makes the approach applicable to optimal design problem with a large number design variable. The accuracy and efficiency of the proposed method are demonstrated by two numerical examples.

8. Acknowledgement

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9. References

[1] Kalman RE, Bertram, JE,Control System Analysis and Design Via the "Second Method" of Lyapunov: I—Continuous-Time Systems. Journal of Fluids Engineering, 82(2), 371-393, 1960.
[2] Ogata K, Yang Y, Modern control engineering, 1970.
[3] Truhar N, Veselic K. An efficient method for estimating the optimal dampers' viscosity for linear vibrating systems using Lyapunov equation. SIAM Journal on Matrix Analysis and Applications, 31(1): 18-39, 2009.
[4] Kuzmanović I, Tomljanović Z, Truhar N. Optimization of material with modal damping. Applied mathematics and computation, 218(13): 7326-7338, 2012.
[5] Cox S J, Nakić I, Rittmann A, et al. Lyapunov optimization of a damped system. Systems & control letters, 53(3): 187-194, 2004.
[6] Marano GC, Greco R, Chiaia B, A comparison between different optimization criteria for tuned mass dampers design. Journal of Sound and Vibration, 329(23), 4880-4890, 2010.
[7] Marano GC, Greco R, Trentadue F, Chiaia B, Constrained reliability-based optimization of linear tuned mass dampers for seismic control. International Journal of Solids and Structures, 44(22), 7370-7388, 2007.
[8] Chang W, Gopinathan S V, Varadan V V, et al. Design of robust vibration controller for a smart panel using finite element model. Journal of vibration and acoustics, 124(2): 265-276, 2002.
[9] Wang BP, Kitis L, Pilkey WD, Transient Response Optimization of Vibrating Structures by Liapunov's Second Method. J. Sound Vib., 96, pp. 505–512, 1984.
[10] Du D, Analytical solutions for DVA optimization based on the Lyapunov equation. Journal of Vibration and Acoustics, 130(5), 054501, 2008.