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Optimization of Elastic Plastic Plates of Piecewise Constant Thickness

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1. Abstract
Numerical and analytical methods of analysis and optimization of elastic plastic plates are developed. The cases of linear and non-linear yield surfaces are studied. Necessary optimality conditions are derived with the aid of variational methods. The obtained system of equations is solved numerically. The effectivity of the design established is assessed in the cases of one- and multi-stepped plates made of Tresca or Mises materials.
2. Keywords: plate, optimization, elastic plastic material, minimum weight.

3. Introduction

Evidently, there exists the need for new computer-aided techniques for calculation and optimization of elastic plastic plates. Optimization of axisymmetric plates operating in the range of elastic plastic deformations was studied by Lellep and Polikarpus [2, 3] in the case of the material obeying Tresca's yield condition and by Lellep and Vlassov [4, 5] in the case of von Mises material.
New analytical and numerical techniques of optimization of axisymmetric plates are developed in the present paper. The material of plates is an ideal elastic plastic material obeying a linear or non-linear yield condition and the associated flow law.

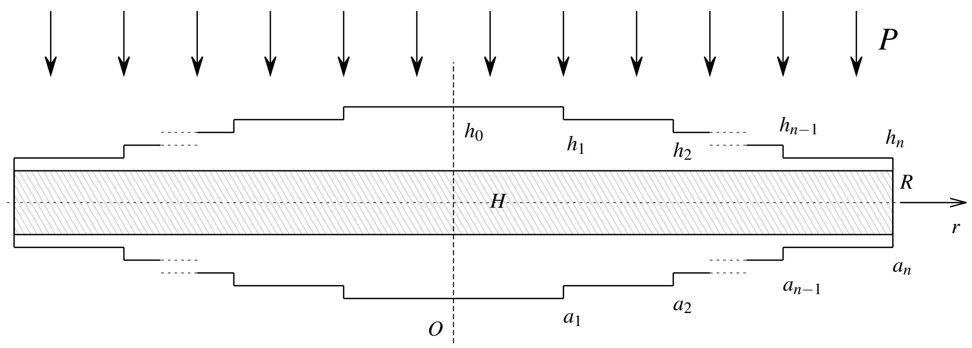

Figure 1: Stepped plate.

4. The cost function

Circular and annular plates with radii R (outer radius) and a (inner one) will be considered. Let us assume that an axisymmetric plate is subjected to the axisymmetric transverse pressure of intensity $p = p(r)$, where r is the current radius. The analysis will be carried out under the assumption that the hypotheses of Kirchhoff hold good in the regimes of elastic and plastic deformations. The plates with sandwich cross section will be considered. A sandwich plate is a structure which consists of two carrying layers of thickness h and of a layer of a core material between the rims. Let the thickness of carrying layers be piecewise constant, e. g.

$$h = h_j, \quad r \in S_j \tag{1}$$

where $S_j = (a_j, a_{j+1})$; $j = 0, \ldots, n$.
Evidently, the plate can be subdivided into elastic and plastic regions in the case of a sandwich plate. Let us denote the elastic region by S_e and the plastic region by S_p. In principle, both of these may consist of several regions S_{ej} and S_{pj}, respectively. Here $S_{ej} = S_j$ if the region S_j is a pure elastic one, e. g. if $j \in K_e$. Similarily, $S_{pj} = S_j$,

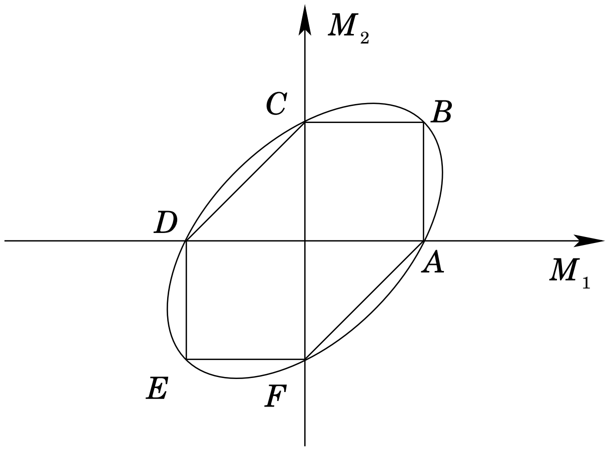

Figure 2: Yield conditions.

if $j \in K_p$. These intervals where both, the elastic and plastic deformations take place are denoted by S_{ep} whereas $S_{ep} = S_j$ for $j \in K_{ep}$. Evidently,

$$K_e \cup K_p \cup K_{ep} = \{0, 1, \ldots, n\}. \tag{2}$$

The thickness of the rim is much smaller than the thickness of the core material H. Note that the quantities h_0, h_j and a_j $(j = 1, \ldots, n)$ are preliminarily unknown constant design parameters when solving a problem of optimization.
As regards the formulation of an optimization problem, one can find in literature a lot of different particular problems (see Lellep [1]).
In the general case the cost function can be presented as

$$J = \sum_{j=0}^{n} \left(G_j + \int_{S_j} F_j dr \right) \tag{3}$$

where G_j and F_j are given functions of design parameters. For the sake of simplicity it is assumed that the functions F_j and G_j are continuous and continuously differentiable with respect to their variables.
The optimal design to be established must satisfy the constraints imposed on the stress-strain state of the plate. Let the additional constraints be presented as integral constraints

$$\sum_{j=0}^{n} \int_{S_j} F_{ij}^0 dr \leq K_i \tag{4}$$

for $i = 1, \ldots, k$. The functions F_{ij}^0 in Eq. (4) are given continuous functions of design parameters and K_i – given constants.
In particular cases the optimization problem consists in the determination of design parameters so that the cost function (total weight, for instance) attains its minimum value and the constraints imposed on the stress strain state of the plate are satisfied. For instance, the cost of carrying layers can be presented in the form

$$V = \sum_{j=0}^{n} h_j (a_{j+1}^2 - a_j^2) \tag{5}$$

where $a_0 = a$ and $a_{n+1} = R$.

5. Basic equations

Making use of the classical plate theory the principal moments M_1, M_2 and the shear force Q are connected as

$$\begin{aligned} &\frac{d}{dr}(rM_1) - M_2 - rQ = 0, \\ &\frac{d}{dr}(rQ) + Pr = 0. \end{aligned} \tag{6}$$

Equations (6) hold good equally in elastic and plastic regions of a plate.
In an elastic region of the plate the Hooke's law holds good. According to the generalized Hooke's law

$$\begin{aligned} M_1 &= D_j(\kappa_1 + \nu\kappa_2), \\ M_2 &= D_j(\kappa_2 + \nu\kappa_1) \end{aligned} \tag{7}$$

where ν is the Poisson ratio. In Eq. (7) and henceforth

$$D_j = \frac{EH^2 h_j}{2(1-\nu^2)} \tag{8}$$

where E stands for the Young modulus. Here κ_1 and κ_2 stand for the curvatures (W is the transverse deflection)

$$\begin{aligned} \kappa_1 &= -\frac{d^2W}{dr^2}, \\ \kappa_2 &= -\frac{1}{r}\frac{dW}{dr}. \end{aligned} \tag{9}$$

It can be shown that the system of governing equations in an elastic region for $r \in S_{ej}$ can be presented as (see Lellep, Vlassov [4, 5])

$$\begin{aligned} \frac{dW}{dr} &= Z, \\ \frac{dZ}{dr} &= -\frac{M_1}{D_j} - \frac{\nu Z}{r}, \\ \frac{dM_1}{dr} &= -\frac{(1-\nu^2)D_j Z}{r^2} - \frac{(1-\nu)M_1}{r} + Q \end{aligned} \tag{10}$$

whereas the following equation

$$M_2 - \nu M_1 + \frac{D_j(1-\nu^2)Z}{r} = 0 \tag{11}$$

must be satisfied for each $r \in S_{ej}$. Note that here $j \in K_e$ and $j \in K_{ep}$ and Z can be treated as an auxiliary variable.
In plastic regions of the plate the stress profile lies on a yield surface $\Phi = 0$. It is assumed that the material of the plate obeys a yield condition

$$\Phi_j(M_1, M_2, M_0) \leq 0 \tag{12}$$

for $r \in S_{pj}$, $j \in K_p$.
It can be shown that in the case of a von Mises material one can take

$$\Phi_j = M_1^2 - M_1 M_2 + M_2^2 - M_{0j}^2. \tag{13}$$

In the case of a Tresca plate one has to check the suitability of each side of the hexagon $ABCDEF$ (Fig. 2) separately. However, if it is clear previously that, for instance, $M_1 \geq 0$, $M_2 \geq 0$ for each $r \in S_p$ one can concentrate at the flow regime BC (AB is not suitable for most cases). Thus now

$$\Phi_j = M_2 - M_{0j} \tag{14}$$

for $r \in S_{pj}$, $j \in K_p$. The associated flow law states that

$$\begin{aligned} \kappa_1 &= \frac{\lambda \partial \Phi_j}{\partial M_1}, \\ \kappa_2 &= \frac{\lambda \partial \Phi_j}{\partial M_2} \end{aligned} \tag{15}$$

for $r \in S_{pj}$, $j \in K_p$. Here λ is a non-negative scalar multiplier. Thus combining Eq. (12), (15) with Eq. (6) one has

$$\begin{aligned} \frac{dW}{dr} &= Z, \\ \frac{dZ}{dr} &= \frac{Z\frac{\partial \Phi_j}{\partial M_1}}{r\frac{\partial \Phi_j}{\partial M_2}}, \\ \frac{dM_1}{dr} &= \frac{M_2}{r} - \frac{M_1}{r} + Q, \end{aligned} \tag{16}$$

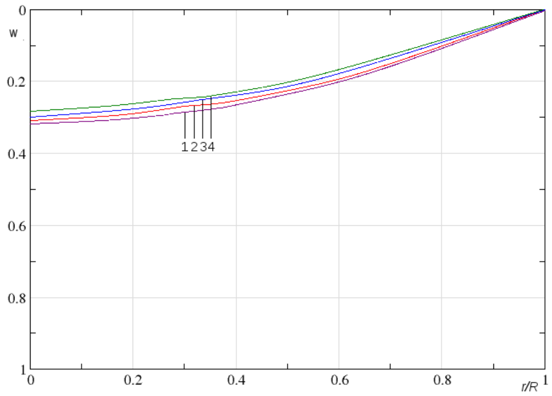

Figure 3: Transverse deflection.

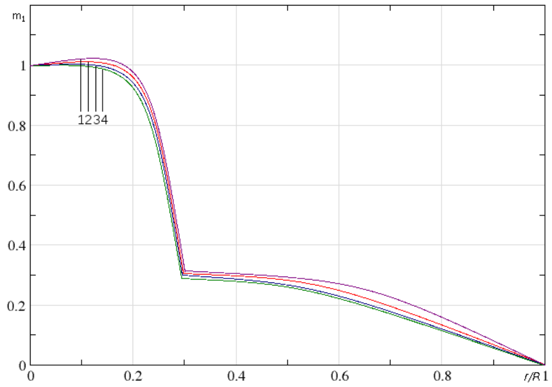

Figure 4: Bending moment.

where the quantity Q can be handled as a given function. Indeed it follows from the equilibrium equations that

$$Q = -\frac{1}{r}\int_a^r P(r) r dr. \tag{17}$$

6. Necessary optimality conditions

In order to derive optimality conditions let us introduce an extended functional

$$\begin{aligned} J_* = {} & \sum_{j=0}^{n} G_j + \sum_{j\in K_{ej}} \int_{S_{ej}} \left\{ \psi_1 \left(\frac{dW}{dr} - Z \right) + \psi_2 \left(\frac{dW}{dr} + \frac{M_1}{D_j} + \frac{\nu Z}{r} \right) \right. \\ & \left. + \psi_3 \left(\frac{dM_1}{dr} + (1-\nu^2) D_j + \frac{(1-\nu) M_1}{r} - Q \right) + \nu_{0j} \left(M_2 + \frac{D_j (1-\nu^2) Z}{r} - \nu M_1 \right) \right\} dr \\ & + \sum_{j\in K_{pj}} \int_{S_{pj}} \left\{ \psi_1 \left(\frac{dW}{dr} - Z \right) + \psi_2 \left(\frac{dZ}{dr} - \frac{Z \frac{\partial \Phi_j}{\partial M_1}}{r \frac{\partial \Phi_j}{\partial M_2}} \right) + \psi_3 \left(\frac{dM_1}{dr} - \frac{M_2}{r} + \frac{M_1}{r} - Q \right) \right\} dr \\ & + \sum_{j=0}^{n} \int_{a_j}^{a_{j+1}} [\phi_{0j} (\phi_j + \Theta_j^2) + F_i + \nu_i F_{ij}^0] dr \end{aligned} \tag{18}$$

where Θ_j are new control functions and $\psi_1 - \psi_3$ – the adjoint variables. The boundary conditions are not presented

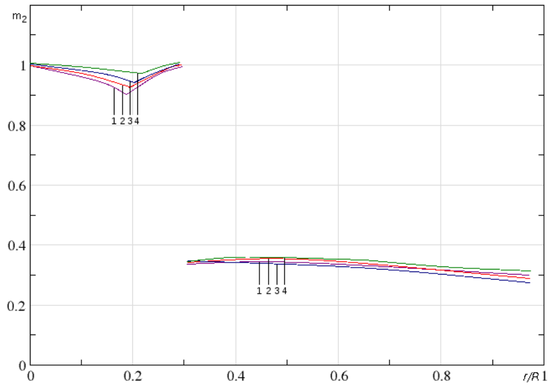

Figure 5: Circumferential moment.

in Eq. (18); these must be taken into account when solving the equation $\Delta J_* = 0$. Calculating the total variation of Eq. (18) and equalizing it to zero leads to the set of optimality conditions. The procedure of the variation of Eq. (18) is similar to that performed in [4, 5].

7. Numerical results

In the case of a non-linear materials the problem is solved numerically making use of finite elements and the method of wavelets. The results of calculations are presented in Fig. 3–5 in the case when the number of steps $n = 1$.

The distributions of transverse deflections, radial and circumferential bending moments are presented in Fig. 3, 4 and Fig. 5, respectively. Different curves in Fig. 3–5 correspond to the plates made of a Hill material and subjected to the uniform transverse pressure. Here the labels of curves correspond to $p = 3.15$; $p = 3.25$; $p = 3.45$ and $p = 3.65$, respectively, and

$$p = \frac{PM_{00}}{R^2}, \qquad \alpha_i = \frac{a_i}{R}, \qquad m_i = \frac{M_i}{M_{00}}. \tag{19}$$

The results depicted in Fig. 3–5 correspond to the case when $h_1 = 0.6h_0$; $a_1 = 0.35R$. The region of plastic deformations reaches to $r = 0.304$ for $p = 3.15$ and to $r = 0.385$ for $p = 3.65$.

The distributions of M_1 and M_2 are depicted in Fig. 6–7 in the case of the Tresca material. It can be seen from Fig. 6, 7 that the hoop moment is discontinuous and the radial moment is a continuous non-smooth, as might be expected.

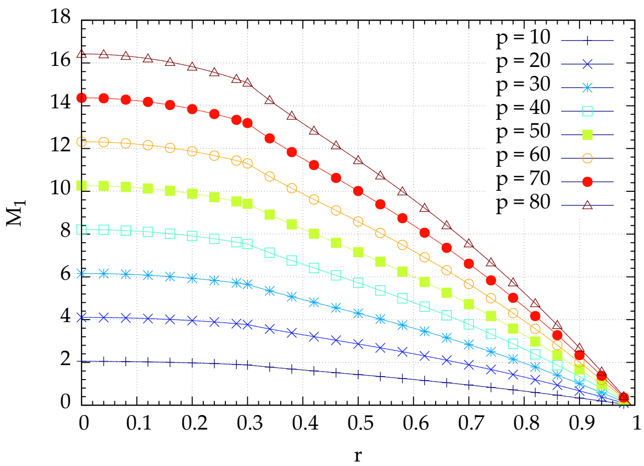

Figure 6: Bending moment.

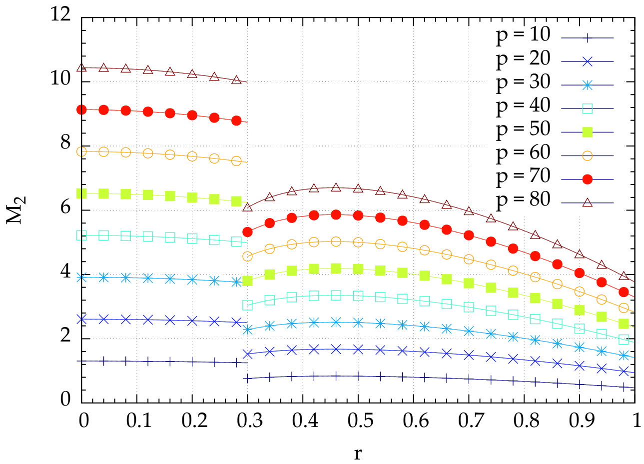

Figure 7: Circumferential moment.

8. Concluding remarks

Methods of optimal design of axisymmetric plates are developed under the assumption that the plates are operating in the elastic plastic stage of deflection. Necessary conditions of optimality are derived making use of variational methods. Numerical results are obtained for plates obeying the Mises or Tsai-Wu criterion with the aid of Haar wavelets. Calculations carried out showed remarkable material saving can be achieved when using the design of stepped plate.

9. Acknowledgements

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