Structural Optimization under Complementarity Constraints

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1. Abstract
This paper provides an overview of the challenging class of structural optimization problems with complementarity conditions, generally known as a "mathematical program with equilibrium constraints" (MPEC). Complementarity, mathematically defined by the perpendicularity of two sign constrained vectors, describes such common mechanical behaviour as elastoplasticity and contact conditions. The MPEC is in effect the inverse counterpart of a state problem formulated as a "mixed complementarity problem" (MCP), and is moreover far more challenging to process since an MPEC is in general nonsmooth and nonconvex. We briefly describe a promising class of solution methods, all based on some regularization technique, to convert the MPEC into a standard nonlinear programming (NLP) problem, and illustrate its application for the optimal design of engineering structures.

2. Keywords: complementarity, nonconvex and nonsmooth mathematical program, structural optimization.

3. Introduction
Complementarity (the requirement that two nonnegative vectors are orthogonal) is a typical and recurrent mathematical feature in the nonlinear analysis of structures, e.g. to represent elastoplasticity and contact-like conditions. The resulting state problems lead to instances of mathematical programs known generally as "mixed complementarity problems" (MCPs) [1] for which, under certain conditions (e.g. definiteness of some key matrices), can be efficiently solved. However, the inverse problem that for example arises in structural optimization under complementarity conditions is far more challenging to process since the underlying mathematical programming problem, known as a "mathematical program with equilibrium constraints" (MPEC) [2], is nonsmooth and/or nonconvex.
We introduce the concept of complementarity, and review the state problem and its solution before presenting the generic formulation for structural optimization under complementarity constraints [3-7]. We then provide an overview of a promising class of solution methods that can be used to solve the resulting MPECs [2]. These all involve application of some regularizing technique followed by conversion of the MPEC into a standard nonlinear programming (NLP) problem. Finally, we give two illustrative examples to illustrate this approach.

4. Complementarity Conditions in Engineering Mechanics
Various engineering state problems can be formulated as a standard MCP [1] which, in general, consists of three pieces of basic information, namely lower bounds $\mathbf{z}_l \in \Re^l$, upper bounds $\mathbf{z}_u \in \Re^l$ and functions $\mathbf{Y}(\mathbf{z}) \in \Re^l$. The aim is to

$$
\begin{aligned}
&\text{find} && \mathbf{z}, \mathbf{v}, \mathbf{k} \in \Re^l \\
&\text{subject to} && \mathbf{Y}(\mathbf{z}) - \mathbf{v} + \mathbf{k} = \mathbf{0} \\
& && \mathbf{z}_l \le \mathbf{z} \le \mathbf{z}_u \\
& && \mathbf{v}^{\mathrm{T}}(\mathbf{z} - \mathbf{z}_l) = \mathbf{0},\ \mathbf{v} \ge \mathbf{0} \\
& && \mathbf{k}^{\mathrm{T}}(\mathbf{z}_u - \mathbf{z}) = \mathbf{0},\ \mathbf{k} \ge \mathbf{0}
\end{aligned}
\tag{1}
$$

where $-\infty \le \mathbf{z}_l \le \mathbf{z}_u \le \infty$ and $\mathbf{Y}(\mathbf{z})$ are continuously differentiable. As an illustration, we briefly review in the following two engineering mechanics problems (see e.g. [7]) formulated as MCP (1); one involves elastoplasticity, the other contact conditions with complementarity conditions schematically shown in Fig. 1.

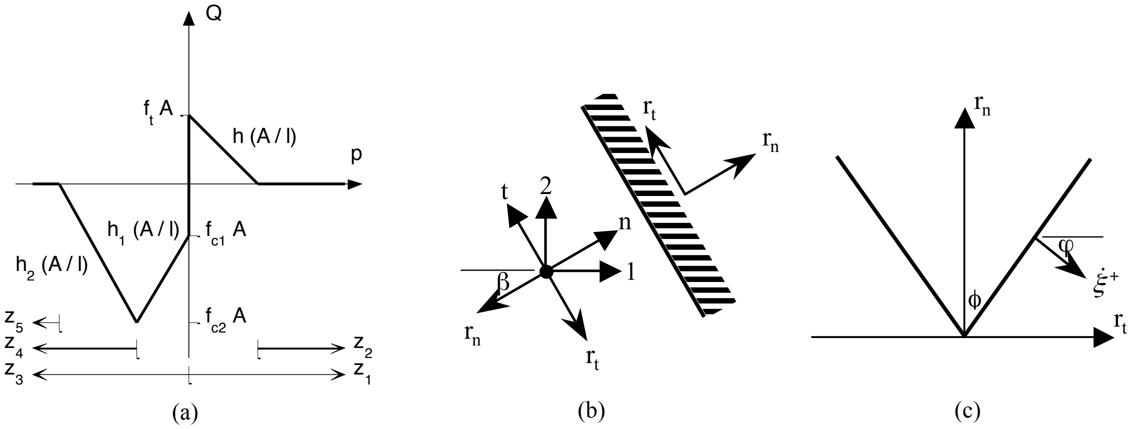

Figure 1: Complementarity conditions (a) softening material, (b) contact forces, (c) friction law.

4.1 Analysis of Elastoplastic Structures

The governing state problem for the holonomic (path-independent) analysis of structures (suitably discretised into n number of elements, d degree of freedoms, m generalized stresses/strains and y yield functions) with inelastic material properties can be cast as an MCP in variables (**Q**,**u**,**z**) as follows [4]:

$$\begin{aligned} &-\alpha\mathbf{f} + \mathbf{C}^{\mathrm{T}}\mathbf{Q} = \mathbf{0} \\ &\mathbf{Q} - \mathbf{SCu} + \mathbf{SNz} = \mathbf{0} \\ &\mathbf{w} = -\mathbf{N}^{\mathrm{T}}\mathbf{Q} + \mathbf{Hz} + \mathbf{r} \geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0},\ \mathbf{w}^{\mathrm{T}}\mathbf{z} = 0 \end{aligned} \tag{2}$$

Clearly problem (2) is an instance of MCP (1), where the first two relations in Eq.(2) correspond to the first condition in Eq.(1) with $\mathbf{0} \leq \mathbf{z} \leq \infty$, $\mathbf{v} = \mathbf{w} \geq \mathbf{0}$ and $\mathbf{k} = \mathbf{0}$.

Physically, the first equation in MCP (2) describes linear equilibrium between the externally applied forces $\alpha\mathbf{f} \in \Re^d$ and the generalized stresses $\mathbf{Q} \in \Re^m$ through a constant compatibility matrix $\mathbf{C} \in \Re^{m\times d}$, where α and $\mathbf{f} \in \Re^d$ denote a positive load scalar and a basic force vector, respectively. The second equation expresses the relationship between stresses **Q** and the elastic strains defined as $\mathbf{q} - \mathbf{p} \in \Re^m$, where $\mathbf{S} \in \Re^{m\times m}$, $\mathbf{q} \in \Re^m$ and $\mathbf{p} \in \Re^m$ are, respectively, the conventional (unassembled) elastic stiffness matrix, generalized strains written in terms of nodal displacements $\mathbf{u} \in \Re^d$, and generalized plastic strains. In the third and final relation, an associative flow rule prescribes the plastic strains **p** as functions of plastic multipliers $\mathbf{z} \in \Re^y$, where $\mathbf{N} \in \Re^{m\times y}$ collects the normal directions to all piecewise linear (PWL) yield hyperplanes [8] in Fig. 1a. The yield functions $\mathbf{w} \in \Re^y$ mathematically describe the PWL yield model of this Fig. 1a in terms of **Q** and **z**, where $\mathbf{H} \in \Re^{y\times y}$ and $\mathbf{r} \in \Re^y$ denote a hardening/softening matrix and a plastic limit vector, respectively. Finally, the complementarity condition $\mathbf{w}^{\mathrm{T}}\mathbf{z} = 0$ (describing a componentwise relationship $w_j \geq 0$, $z_j \geq 0$ and $w_j z_j = 0$ for $j = 1$ to y) between the two positive sign constrained vectors $\mathbf{w} \geq \mathbf{0}$ and $\mathbf{z} \geq \mathbf{0}$ implies either elastic ($w_j > 0$ and $z_j = 0$) or plastic ($w_j = 0$ and $z_j > 0$) behaviour, and also allows reversal (holonomy) of plastic strains

An elastoplastic analysis maps out the complete load versus displacement responses of the inelastic structure by collecting the resulting variables **Q**, **u** and **z** obtained from a series of MCP (2) solves under specified increasing values of load multiplier α. MCP (2) can be processed directly using, for instance, the industry standard complementarity solver PATH [1]. For computational and modelling convenience PATH is often called from within some mathematical programming environment such as GAMS (an acronym for "general algebraic modelling system") [9].

4.2 Analysis of Structures with Frictional Contacts

We consider rigid perfectly plastic structures with c unilateral frictional contacts as shown in Figs. 1b-c. The state problem can be formulated as the following MCP in variables (α, $\mathbf{Q}$, $\dot{\mathbf{u}}$, $\dot{\mathbf{z}}$, $\mathbf{r}_n$, $\mathbf{r}_t$, $\dot{\boldsymbol{\xi}}$) [7]:

$$
\begin{aligned}
&\mathbf{f}^{\mathrm{T}}\dot{\mathbf{u}} = 1 \\
&-\alpha\mathbf{f} + \mathbf{C}^{\mathrm{T}}\mathbf{Q} + \mathbf{C}_n^{\mathrm{T}}\mathbf{r}_n + \mathbf{C}_t^{\mathrm{T}}\mathbf{r}_t = \mathbf{0} \\
&-\mathbf{C}\dot{\mathbf{u}} + \mathbf{N}\dot{\mathbf{z}} = \mathbf{0} \\
&-\mathbf{C}_t\dot{\mathbf{u}} + \mathbf{V}_t\dot{\boldsymbol{\xi}} = \mathbf{0} \\
&\mathbf{w} = -\mathbf{N}^{\mathrm{T}}\mathbf{Q} + \mathbf{r} \geq \mathbf{0},\ \dot{\mathbf{z}} \geq \mathbf{0},\ \mathbf{w}^{\mathrm{T}}\dot{\mathbf{z}} = 0 \\
&\boldsymbol{\pi}_c = \mathbf{N}_n^{\mathrm{T}}\mathbf{r}_n - \mathbf{N}_t^{\mathrm{T}}\mathbf{r}_t \geq \mathbf{0},\ \dot{\boldsymbol{\xi}} \geq \mathbf{0},\ \boldsymbol{\pi}_c^{\mathrm{T}}\dot{\boldsymbol{\xi}} = 0 \\
&\boldsymbol{\pi}_n = -\mathbf{C}_n\dot{\mathbf{u}} + \mathbf{V}_n\dot{\boldsymbol{\xi}} \geq \mathbf{0},\ \mathbf{r}_{\mathrm{n}} \geq \mathbf{0},\ \boldsymbol{\pi}_n^{\mathrm{T}}\mathbf{r}_n = 0
\end{aligned}
\tag{3}
$$

Solution of this MCP (3) provides one set of response variables α ,$\mathbf{Q}$, $\dot{\mathbf{u}}$, $\dot{\mathbf{z}}$,$\mathbf{r}_n$,$\mathbf{r}_t$, $\dot{\boldsymbol{\xi}}$ for the structural system considering rigid perfectly plastic material properties and nonassociative frictional contact conditions.

The first relation represents a (normalised) positive dissipation produced by $\mathbf{f}$ and displacement rates $\dot{\mathbf{u}} \in \Re^d$. Linear equilibrium between $\alpha\mathbf{f}$, $\mathbf{Q}$ and the two contact forces in the normal $\mathbf{r}_n \in \Re^c$ and tangential $\mathbf{r}_t \in \Re^c$ directions is given in the second relation, where $\mathbf{C}_n, \mathbf{C}_t \in \Re^{c \times d}$ are the corresponding compatibility matrices at the contacts. Compatibility between $\dot{\mathbf{u}}$ and the plastic multiplier rates $\dot{\mathbf{z}} \in \Re^y$ is described by the third relation. The fourth relation indicates the compatibility between tangential displacement rates $\mathbf{C}_t\dot{\mathbf{u}}$ along the contact interface and the sliding rates $\mathbf{V}_t\dot{\boldsymbol{\xi}}$, where $\dot{\boldsymbol{\xi}} \in \Re^{2c}$ and $\mathbf{V}_t \in \Re^{c \times 2c}$ are sliding multiplier rates and a constant matrix, respectively. Finally, the three complementarity conditions between (i) $\mathbf{w}$ and $\dot{\mathbf{z}}$, (ii) $\boldsymbol{\pi}_c \in \Re^{2c}$ and $\dot{\boldsymbol{\xi}}$, and (iii) $\boldsymbol{\pi}_n \in \Re^c$ and $\mathbf{r}_n$ enforce a perfectly plastic material law, the assumed frictional contact model (shown in Figs. 1b-c) and nonpenetration at contact interfaces, respectively [7]. $\mathbf{V}_n \in \Re^{c \times 2c}$, $\mathbf{N}_n \in \Re^{c \times 2c}$ and $\mathbf{N}_t \in \Re^{c \times 2c}$ are appropriate transformation matrices.

5. Optimization with Complementarity Conditions

We now consider the inverse problems corresponding to MCPs (2) and (3) that arise in the optimal design of structures with inelastic material properties and/or frictional contact conditions. The aim of such a design is to automatically determine the minimum and safe material distribution (i.e. typically represented by unknown cross-sectional areas $\mathbf{A}$) of the structural members such that the predefined physical and material requirements are simultaneously satisfied. This involves the formulations and solutions of "nonstandard" optimization problems, known as MPECs [2], where the so-called "equilibrium constraints" are, in our case, complementarity constraints expressing certain intrinsic structural behaviors, such as the ones in MCPs (2) and (3).

5.1 Optimization of Elastoplatic Structures

The MPEC in variables ($\mathbf{A}$,$\mathbf{Q}$,$\mathbf{u}$,$\mathbf{z}$) that describes the optimal design of elastoplastic (softening) structures is [4]

$$
\begin{aligned}
\text{Minimize}\quad & V(\mathbf{A}) \\
\text{Subject to}\quad & -\alpha\mathbf{f} + \mathbf{C}^{\mathrm{T}}\mathbf{Q} = \mathbf{0} \\
& \mathbf{Q} - \mathbf{S}(\mathbf{A})\mathbf{C}\mathbf{u} + \mathbf{S}(\mathbf{A})\mathbf{N}\mathbf{z} = \mathbf{0} \\
& \mathbf{w} = -\mathbf{N}(\mathbf{A})^{\mathrm{T}}\mathbf{Q} + \mathbf{H}(\mathbf{A})\mathbf{z} + \mathbf{r}(\mathbf{A}) \geq \mathbf{0},\ \mathbf{z} \geq \mathbf{0},\ \mathbf{w}^{\mathrm{T}}\mathbf{z} = 0 \\
& \mathbf{A}_{lo} \leq \mathbf{A} \leq \mathbf{A}_{up} \\
& \text{Technological and displacement constraints}
\end{aligned}
\tag{4}
$$

MPEC (4) minimizes the total weight/volume $V(\mathbf{A})$ of the structure (directly related to the total cost) subject to the constraints given by the state problem in MCP (2), where the stiffness matrix $\mathbf{S}$, the normality matrix $\mathbf{N}$, the softening/hardening matrix $\mathbf{H}$ and the vector of yield limits $\mathbf{r}$ are written in terms of the unknown cross sectional areas $\mathbf{A}$ that are bounded within available lower $\mathbf{A}_{lo}$ and upper $\mathbf{A}_{up}$ size limits. Technological and displacement constraints [3,4] impose specific conditions to accommodate, for instance, the requirement of identical member sizes for certain groups of structural members and displacement limits at some specified locations, respectively.

5.2 Optimization of Structures with Frictional Contacts

The inverse or optimal design problem to MCP (3) aims to obtain a minimum volume solution for rigid perfectly plastic structures with frictional contacts. The governing MPEC formulation in variables ($\mathbf{A}$,$\mathbf{Q}$, $\dot{\mathbf{u}}$, $\dot{\mathbf{z}}$,$\mathbf{r}_n$,$\mathbf{r}_t$, $\dot{\boldsymbol{\xi}}$) [3] is

$$
\begin{aligned}
&\text{Minimize} && V(\mathbf{A}) \\
&\text{Subject to} && \mathbf{f}^{\mathrm{T}}\dot{\mathbf{u}} = 1 \\
& && -\alpha\mathbf{f} + \mathbf{C}^{\mathrm{T}}\mathbf{Q} + \mathbf{C}_n^{\mathrm{T}}\mathbf{r}_n + \mathbf{C}_t^{\mathrm{T}}\mathbf{r}_t = \mathbf{0} \\
& && -\mathbf{C}\dot{\mathbf{u}} + \mathbf{N}\dot{\mathbf{z}} = \mathbf{0} \\
& && -\mathbf{C}_t\dot{\mathbf{u}} + \mathbf{V}_t\dot{\boldsymbol{\xi}} = \mathbf{0} \\
& && \mathbf{w} = -\mathbf{N}(\mathbf{A})^{\mathrm{T}}\mathbf{Q} + \mathbf{r}(\mathbf{A}) \geq \mathbf{0},\ \dot{\mathbf{z}} \geq \mathbf{0},\ \mathbf{w}^{\mathrm{T}}\dot{\mathbf{z}} = 0 \\
& && \boldsymbol{\pi}_c = \mathbf{N}_n^{\mathrm{T}}\mathbf{r}_n - \mathbf{N}_t^{\mathrm{T}}\mathbf{r}_t \geq \mathbf{0},\ \dot{\boldsymbol{\xi}} \geq \mathbf{0},\ \boldsymbol{\pi}_c^{\mathrm{T}}\dot{\boldsymbol{\xi}} = 0 \\
& && \boldsymbol{\pi}_n = -\mathbf{C}_n\dot{\mathbf{u}} + \mathbf{V}_n\dot{\boldsymbol{\xi}} \geq \mathbf{0},\ \mathbf{r}_{\mathrm{n}} \geq \mathbf{0},\ \boldsymbol{\pi}_n^{\mathrm{T}}\mathbf{r}_n = 0 \\
& && \mathbf{A}_{lo} \leq \mathbf{A} \leq \mathbf{A}_{up} \\
& && \text{Technological constraints}
\end{aligned}
\tag{5}
$$

in which matrix **N** and vector **r** are functions of the area variables **A**.

5.3 MPECs – An Overview

The systematic study of MPECs has increasingly attracted research interest due to the fact that, in addition to being theoretically difficult and computationally challenging, MPECs find numerous applications in economic and engineering problems involving equilibrium systems [10]. An MPEC is an optimization problem, in which some or all constraints are defined by a parametric variational inequality or complementarity system [2]. The most prominent feature of an MPEC, and one that distinguishes it from a standard nonlinear programming (NLP) problem, is the presence of complementarity constraints. These constraints classify the MPEC as a nonlinear disjunctive (or piecewise) program. Consequently, besides the common issues associated with general NLP problems, the MPEC carries with it a "combinatorial curse" – a standard feature of all disjunctive problems.

There are three main reasons why an MPEC is difficult to solve [2]. First, the complementarity constraints are disjunctive. As is well-known from the integer programming literature, disjunctive constraints such those embodied by the complementarity condition (e.g. either $w_j = 0$ or $z_j = 0$) are very difficult to handle. This, as a result, makes the MPEC disjunctive. There is no feasible point for which all inequalities are strictly satisfied. Even under restrictions, this makes the feasible region a union of finitely many closed sets. Second, the feasible region of an MPEC may not be convex. Third, the feasible region of an MPEC may not be connected.

Any subset of these three difficulties may (and frequently) occur making the problem hard to handle and is often expected to show up as a severe numerical instability. To date, no algorithm has yet been proposed to guarantee solution of general MPECs.

6. Regularization Approaches

A direct attempt to solve the MPEC given in Eq. (4) or (5) is likely to suffer from numerical difficulties. A far better approach is to reformulate it as a standard NLP problem by suitably "treating" the complementarity constraints by some regularization technique. The idea is to solve a series of NLP subproblems such that the original complementarity condition is increasingly enforced, as some (positive) scalar parameter μ is increased or decreased. We outline three such NLP-based algorithms in the following.

Penalization: The complementarity term is transferred to the objective function and penalized (e.g. [3,4]). In particular, this involves modifying the objective function by adding the term $\mu\mathbf{w}^{\mathrm{T}}\mathbf{z}$ in MPEC (4) and $\mu(\mathbf{w}^{\mathrm{T}}\dot{\mathbf{z}} + \boldsymbol{\pi}_c^{\mathrm{T}}\dot{\boldsymbol{\xi}} + \boldsymbol{\pi}_n^{\mathrm{T}}\mathbf{r}_n)$ in MPEC (5). The algorithm then simply increases parameter μ at each NLP iterate, with the intention of driving the complementarity term to zero.

Smoothing: The complementarity conditions are replaced by a set of smooth functions $\psi_\mu(w_j, z_j) = 0$ for all j in MPEC (4), and by $\psi_\mu(w_j, \dot{z}_j) = 0$, $\psi_\mu(\pi_{c,j}, \dot{\xi}_{c,j}) = 0$ and $\psi_\mu(\pi_{n,j}, r_{n,j}) = 0$ for all j in MPEC (5). A common function φ_μ used is the Fischer-Burmeister function [11] written as

$$\psi_\mu(w_j, z_j) = \sqrt{w_j^2 + z_j^2 + 2\mu} - (w_j + z_j) \tag{6}$$

This function ψ_μ has the property that $\psi_\mu(w_j, z_j) = 0$ if and only if $w_j \geq 0$, $z_j \geq 0$ and $w_j z_j = 0$. The algorithm then iteratively decreases parameter μ in order to drive the complementarity term to zero (e.g. [7]).

Relaxation: The original complementarity constraints are replaced by their relaxed version $\mathbf{w}^{\mathrm{T}}\mathbf{z} \leq \mu$ in MPEC (4) and by $\mathbf{w}^{\mathrm{T}}\dot{\mathbf{z}} + \boldsymbol{\pi}_c^{\mathrm{T}}\dot{\boldsymbol{\xi}} + \boldsymbol{\pi}_n^{\mathrm{T}}\mathbf{r}_n \leq \mu$ in MPEC (5). The relaxed problem is solved for successively smaller values of μ to force the complementarity term, which is nonnegative at feasible points, to approach zero (e.g. [7]). The success of which algorithm to use can be problem dependent, but we have found that all three regularizations performed robustly for our optimal design problems. The attraction of these schemes is that each subproblem is a standard NLP problem, for which the standard solvers, such as CONOPT [12], can be used.

7. Illustrative Examples

Two examples are provided: one concerns an optimal synthesis involving elastic softening materials (Fig. 1a) [4] and the other rigid perfectly plastic materials with frictional contacts (Figs. 1b-c) [3]. All examples can be solved efficiently by any of the three regularization techniques mentioned above.

The first example considers the simultaneous topology and size design of a 3D cantilever beam (Fig. 2a) subjected to two points loads of 100α and 50α kN ($\alpha = 1$), where v_1 and v_2 denote the corresponding displacements (m). The beam was translationally restrained in all directions at the four corner nodes at its supported end. The displacement limits imposed were $-0.02 \leq v_1, v_2 \leq 0.02$ m. The design adopted a ground structure (shown in Fig. 2b) consisting of truss members. The PWL elastic softening material properties (kN, m units with $E = 28000 \times 10^3$, $f_t = f_{c1} = 14 \times 10^3$, $f_{c2} = 28 \times 10^3$, $h_1 = 16800 \times 10^3$, and $h = h_2 = -2800 \times 10^3$) in Fig. 1a were used throughout, where l defines the member length. Area bounds for all members were set to $0 \leq A \leq \infty$.

The discrete truss model in Fig. 2b contains 99 nodes, 710 members, 285 degrees of freedom and 3550 yield functions.

An optimal design with total volume $V = 0.2305$ m^3 was successfully obtained by solving MPEC (4). The designed member distribution is as drawn in Fig. 2c.

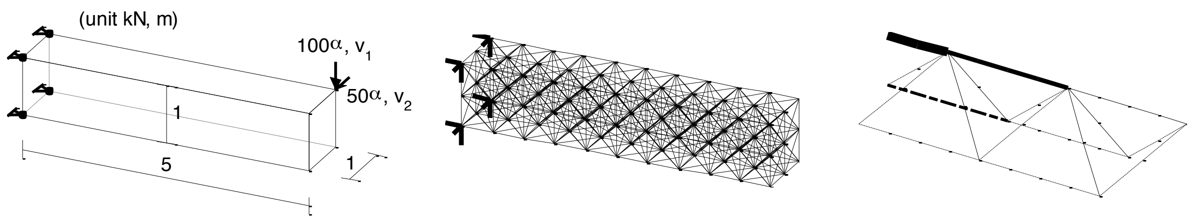

Figure 2: 3D cantilever beam (a) geometry and loads, (b) ground structure, (c) optimal designed structure (solid line denotes tension member and dashed line compression member) [4].

The second example is the 3D double layer roof truss (16 m × 16 m in plan size and 2.828 m in height) shown in Fig. 3a. The structure was restrained only at some bottom layer nodes, in the y-axis direction along its perimeter and in all directions at four corners. At these bottom layer nodes 2, 3, 5, 8, 9, 12, 14 and 15, unilateral (along the x-axis) Coulomb frictional (Fig. 1c with $\tan\phi = 0.3$ and $\varphi = 0$) supports were installed along the perimeter.

The roof truss was designed for $\alpha = 20$ applied at top layer nodes, namely $F(x{:}y{:}z) = (16\alpha{:}4\alpha{:}-16\alpha)$ at each node shown by o in Fig. 3a, and $0.5F$ and $0.25F$ at the nodes indicated by • and ⊗, respectively. Standard CHS sections with a yield stress of $f_y = 250 \times 10^3$ kNm^{-2} were adopted. All bottom layer members had the same area A_1, all top layer members area A_2, and diagonal members area A_3, with all areas bounded as $820 \times 10^{-6} \leq A \leq 2710 \times 10^{-6}$ m^2.

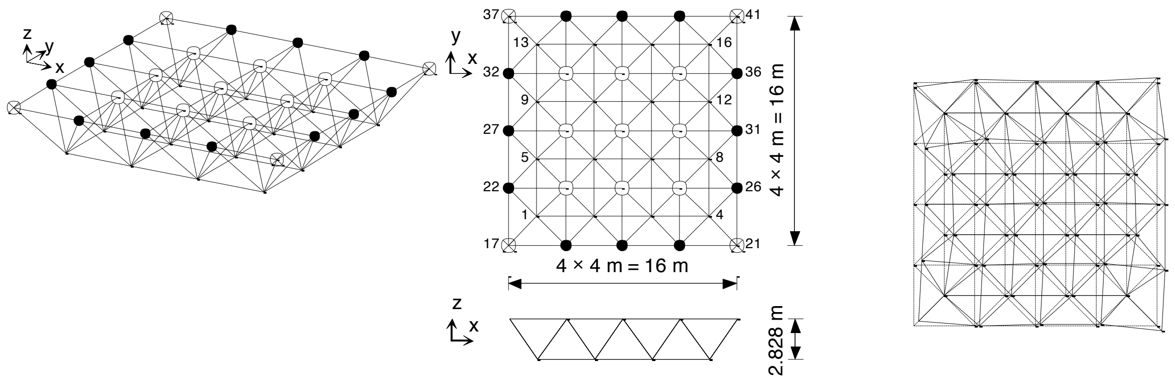

Figure 3: 3D double layer roof truss (a) geometry and loads and (b) collapse mechanism (o, • and ⊗ denote applied forces F, $0.5F$ and $0.25F$, respectively) [3].

The discrete truss model consists of 128 members, 41 nodes and 103 degrees of freedom.
The MPEC (5) was successfully solved to provide the optimal design in Fig. 3b, with $V = 0.8899$ m^3, $A_1 = 2044 \times 10^{-6}$ m^2, $A_2 = 1093 \times 10^{-6}$ m^2 and $A_3 = 2027 \times 10^{-6}$ m^2. The corresponding collapse mechanism in Fig. 3b involves translation at 6 contacts (namely supports 2, 3, 5, 8, 9 and 12) and no translation at 2 contacts (supports 14 and 15).

8. Concluding Remarks

Various state problems in engineering mechanics can be formulated as mathematical programs with complementarity constraints or more specifically as MCPs. The inverse or synthesis problems to such MCPs lead naturally to a challenging problem class known as MPECs. This short review is intended to provide an overview of the formulations and solution approaches to certain MPECs that arise in the context of structural optimization. The complementarity conditions describe naturally and elegantly elastoplasticity and contact conditions.
While MCPs that arise in the structural mechanics context are eminently solvable since most possess key matrices with "nice" properties (e.g. positive definiteness), MPECs, on the other hand, are far more difficult to solve since they can be disjunctive, nonconvex and/or nonsmooth. Such properties are invariably associated with severe computational difficulties, similar to those in integer programming.
In spite of these difficulties, we have had considerable success in solving structural optimization problems formulated as MPECs. The key idea is to regularize the complementarity conditions and transform the MPEC into a standard NLP problem, the iterative solution of which increasingly enforces complementarity. Three such techniques are penalization, smoothing and relaxation. All perform equally well with the structural optimization described. Numerous examples, two of which are provided herein, attest to their robustness and efficiency.

9. Acknowledgement

This research was supported by the Australian Research Council.

10. References

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