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A New Method for Maximum Dynamic Response Topology Optimization in the Time Domain

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1. Abstract
The widely used static topology optimization methods try to find optimal design of structures under loads that are static or vary slowly. However, when the loads change rapidly, their time-dependent characteristic and the inertia effect must be taken into consideration, and therefore dynamic response topology optimization methods should be employed. Two difficulties in solving such problems are the treatment of time-dependent responses and sensitivity analysis. This paper proposes a one-parameter functional to approximate the extreme value of time-dependent response. The accuracy of the approximation can be controlled by the parameter and some of its important properties are discussed. The proposed functional is incorporated into topology optimization problem to minimize the maximum value of time-dependent response at prescribed material volume. The displacement of a specific point of the structure is considered. The density-based approach is used to solve the topology optimization problems and the adjoint variable method is employed to perform sensitivity analysis. The design variables are updated by the Method of Moving Asymptotes. Two numerical examples are conducted to demonstrate the effectiveness of the proposed method and the time-dependent characteristic of the dynamic loads and inertia effect on the topology optimization results.

2. Keywords: Time-dependent response Topology Optimization One-parameter functional Adjoint variable method.

3. Introduction
The widely used static topology optimization methods try to find optimal design of structures under loads that are static or vary slowly[1]. However, when the loads change rapidly, the time-dependent characteristic and inertia effect must be taken into consideration, and therefore dynamic response topology optimization methods should be employed[2]. This work tries to minimize the maximum dynamic response of structure at prescribed material volume. Two difficulties in solving such problems are the treatment of maximum operator in the objective function and sensitivity analysis. This work proposes a one-parameter functional to approximate the maximum value of time-dependent response and incorporated it into topology optimization problem. The density-based approach is used to solve the topology optimization problems and the adjoint variable method is employed to perform sensitivity analysis. The design variables are updated by the Method of Moving Asymptotes. Two numerical examples are conducted to demonstrate the effectiveness of the proposed method and the time-dependent characteristic of the dynamic loads and inertia effect on the topology optimization results.

4. One-parameter Functional to Approximate the Extreme Value of Time-dependent Response
Let $f(t)$ be a continuous function defined on the interval $[0,T]$, M and m respectively its maximum and minimum values . Zhuang and Xiong[3] proposed a functional $\psi(f,\alpha)$ to approximate M based on the discrete form of V_a-approximation[4]

$$\psi(f,\alpha)=\frac{\int_0^T f(t)\alpha^{f(t)}dt}{\int_0^T \alpha^{f(t)}dt} \tag{1}$$

where α is a positive constant and when it tends to positive infinity, $\psi(f,\alpha)$ tends to M.

This work makes a simple transformation $p=ln\alpha$ and obtains the following one-parameter functional.

$$\varphi(f,p)=\frac{\int_0^T f(t)\mathrm{e}^{pf(t)}dt}{\int_0^T \mathrm{e}^{pf(t)}dt} \tag{2}$$

The functional has the following properties:
(i) $m\le\varphi(f,p)\le M$.
(ii) $\lim\limits_{p\to-\infty}\varphi(f,p)=m$, $\lim\limits_{p\to+\infty}\varphi(f,p)=M$.
(iii) $\varphi(f,p)$ is nondecreasing.

The proof of (i) and (ii) is easy and will be omitted here. To prove (iii), we note that

$$\frac{d\varphi}{dp}=\frac{(\int_0^T f^2(t)\mathrm{e}^{pf(t)}dt)(\int_0^T \mathrm{e}^{pf(t)}dt)-(\int_0^T f(t)\mathrm{e}^{pf(t)}dt)^2}{(\int_0^T \mathrm{e}^{pf(t)}dt)^2} \tag{3}$$

Thanks to the Cauchy-Schwarz inequality, we have

$$(\int_0^T f^2(t)\mathrm{e}^{pf(t)}dt)(\int_0^T \mathrm{e}^{pf(t)}dt)\geq(\int_0^T f(t)\mathrm{e}^{pf(t)}dt)^2 \tag{4}$$

so $d\varphi/dp\geq 0$, which implies (iii). It also can be proved that $d\varphi/dp=0$ iff $f(t)$ is a constant function.

According to the properties listed above, $\varphi(f,p)$ can be a good approximation to the maximum or minimum value of the function $f(t)$ when the value of p is sufficiently large or small. More importantly, the accuracy of this approximation can be controlled by adjusting the parameter p. When increasing the value of p, $\varphi(f,p)$ approximates the maximum value of the function better; on the contrary, if the value of p is decreased, $\varphi(f,p)$ approximates the minimum value of the function better.

5. Maximum Dynamic Response Topology Optimization

5.1. Maximum Dynamic Response Topology Optimization Formulation

In order to solve the dynamic response topology optimization problem, the popular density-based approach is employed. The density-based topology optimization approach assigns each element e a density variable η_e and then links its Young's modulus E_e and structural density(mass density) ρ_e with η_e by appropriate interpolation schemes. To prevent the appearance of the localized modes in dynamics analysis, the polynomial interpolation model proposed in[5] will be used.

$$E_e=E_0(15\eta_e^3+\eta_e)/16 \tag{5}$$

$$\rho_e=\rho_0\eta_e \tag{6}$$

where E_0 and ρ_0 respectively are the Young's modulus and structural density of the solid material.

By using the density-based approach, the dynamic response topology optimization problem in the time domain can be formulated as (7). The objective is to minimize the maximum dynamic response of the structures during the loading phase.

$$\begin{aligned}&\min_{\eta}\ \max_{0\leq t\leq T} f(\mathbf{u}(\mathbf{t}),\eta)\\ &s.t.\ \ \mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t)\\ &\qquad g(\eta)=\mathbf{V}(\eta)-\mathbf{V}_{\max}=\sum_{\mathrm{e}=1}^{\mathrm{N}}\eta_{\mathbf{e}}\mathbf{v}_{\mathbf{e}}-\mathrm{V}_{\max}\leq\mathbf{0}\\ &\qquad \mathbf{0}<\eta_{\mathbf{min}}\leq\eta\leq\mathbf{1}\end{aligned} \tag{7}$$

where $f(\mathbf{u}(\mathbf{t}),\eta)$ is the dynamic response of the structure at time t. $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are respectively the global mass, damping and stiffness matrices. N is the number of elements, v_e is the volume of element e and $V_{\max}$ is the prescribed volume of total material. $\eta_{\mathbf{min}}$ is a positive lower bound vector assigned to η to avoid singularity of the stiffness matrix during topology optimization process. Given the value of the design variable vector η and the initial condition of the structure $\mathbf{u}(0)=\mathbf{u}_0,\dot{\mathbf{u}}(0)=\dot{\mathbf{u}}_0$, the displacement vector $\mathbf{u}(t)$, velocity vector $\dot{\mathbf{u}}(t)$ and acceleration vector $\ddot{\mathbf{u}}(t)$ can be obtained by solving the dynamic equilibrium equations.

It is difficult to solve this problem directly, here we replace the objective function by the one-parameter functional (2) and formulate the following topology optimization problem.

$$\begin{aligned}&\min_{\eta}\ \varphi(f,p)\\ &s.t.\ \ \mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t)\\ &\qquad g(\eta)=\mathbf{V}(\eta)-\mathbf{V}_{\max}=\sum_{\mathrm{e}=1}^{\mathrm{N}}\eta_{\mathbf{e}}\mathbf{v}_{\mathbf{e}}-\mathrm{V}_{\max}\leq\mathbf{0}\\ &\qquad \mathbf{0}<\eta_{\mathbf{min}}\leq\eta\leq\mathbf{1}\end{aligned} \tag{8}$$

5.2. Sensitivity Analysis

As we wish to apply a gradient-based optimization algorithm to find the optimal material distribution of the design, the sensitivity of the objective function with respect to the design variables must be evaluated. Considering that the number of design variables is larger than the number of constraints in topology optimization problems, the adjoint variable method[6] is preferred.

We assume that the loads and the initial conditions are independent of the design, that is $\partial\mathbf{f}/\partial\eta_e=\mathbf{0},\partial\mathbf{u_0}/\partial\eta_{\mathbf{e}}=\mathbf{0},\partial\dot{\mathbf{u}}_{\mathbf{0}}/\partial\eta_{\mathbf{e}}=\mathbf{0}$. Then according to the adjoint variable method, when seeking the derivative $\partial J/\partial\eta_e$, it can be augmented with the product of a Lagrangian multipliers $\lambda(t)$ and the derivative of the residual(which is zero at equilibria)

$$
\begin{aligned}
\frac{\partial \varphi}{\partial \eta_e} &= \frac{\partial \varphi}{\partial \eta_e} + \int_0^T \lambda^T \frac{\partial}{\partial \eta_e}(\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} - \mathbf{f})dt \\
&= C_p \int_0^T (1 + pf - p\varphi)\mathrm{e}^{pf}(\frac{\partial f}{\partial \eta_e} + \frac{\partial f}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \eta_e})dt + \int_0^T \lambda^T \frac{\partial}{\partial \eta_e}(\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u})dt \\
&= C_p \int_0^T (1 + pf - p\varphi)\mathrm{e}^{pf}\frac{\partial f}{\partial \eta_e}dt + \int_0^T \lambda^T(\frac{\partial \mathbf{M}}{\partial \eta_e}\ddot{\mathbf{u}} + \frac{\partial \mathbf{C}}{\partial \eta_e}\dot{\mathbf{u}} + \frac{\partial \mathbf{K}}{\partial \eta_e}\mathbf{u})dt \\
&+ C_p \int_0^T (1 + pf - p\varphi)\mathrm{e}^{pf}\frac{\partial f}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \eta_e}dt + \int_0^T \lambda^T(\mathbf{M}\frac{\partial \ddot{\mathbf{u}}}{\partial \eta_e} + \mathbf{C}\frac{\partial \dot{\mathbf{u}}}{\partial \eta_e} + \mathbf{K}\frac{\partial \mathbf{u}}{\partial \eta_e})dt
\end{aligned}
\tag{9}
$$

where $C_p = 1/\int_0^T \mathrm{e}^{pf(t)}dt$.

Twice integrating-by-parts the last term and rearranging yields

$$
\begin{aligned}
\frac{\partial \varphi}{\partial \eta_e} &= C_p \int_0^T (1 + pf - p\varphi)\mathrm{e}^{pf}\frac{\partial f}{\partial \eta_e}dt + \int_0^T \lambda^T(\frac{\partial \mathbf{M}}{\partial \eta_e}\ddot{\mathbf{u}} + \frac{\partial \mathbf{C}}{\partial \eta_e}\dot{\mathbf{u}} + \frac{\partial \mathbf{K}}{\partial \eta_e}\mathbf{u})dt \\
&+ \int_0^T (\frac{\partial \mathbf{u}}{\partial \eta_e})^T(\mathbf{M}\ddot{\lambda} - \mathbf{C}\dot{\lambda} + \mathbf{K}\lambda + C_p(1 + pf - p\varphi)\mathrm{e}^{pf})(\frac{\partial f}{\partial \mathbf{u}})^T dt + [(\frac{\partial \mathbf{u}}{\partial \eta_e})^T(-\mathbf{M}\dot{\lambda} + \mathbf{C}\lambda) + (\frac{\partial \dot{\mathbf{u}}}{\partial \eta_e})^T \mathbf{M}\lambda]\Big|_{t=T}
\end{aligned}
\tag{10}
$$

Since (10) holds for arbitrary $\lambda(t)$, the Lagrangian multipliers can be chosen to eliminate the last two terms of the right-hand side of (10)

$$
\begin{aligned}
&\mathbf{M}\ddot{\lambda} - \mathbf{C}\dot{\lambda} + \mathbf{K}\lambda = -C_p(1 + pf - p\varphi)\mathrm{e}^{pf}(\tfrac{\partial f}{\partial \mathbf{u}})^T, t \in [0, T] \\
&\lambda(T) = \mathbf{0}, \dot{\lambda}(T) = \mathbf{0}
\end{aligned}
\tag{11}
$$

And the derivative $\partial\varphi/\partial\eta_e$ can now be simply given as

$$
\frac{\partial \varphi}{\partial \eta_e} = C_p \int_0^T (1 + pf - p\varphi)\mathrm{e}^{pf}\frac{\partial f}{\partial \eta_e}dt + \int_0^T \lambda^T(\frac{\partial \mathbf{M}}{\partial \eta_e}\ddot{\mathbf{u}} + \frac{\partial \mathbf{C}}{\partial \eta_e}\dot{\mathbf{u}} + \frac{\partial \mathbf{K}}{\partial \eta_e}\mathbf{u})dt
\tag{12}
$$

In order to obtain $\lambda(t)$, we apply the transformation $\mathbf{\Lambda}(s) = \lambda(T - s)$, then Eq. (11) becomes

$$
\begin{aligned}
&\mathbf{M}\ddot{\mathbf{\Lambda}}(s) + \mathbf{C}\dot{\mathbf{\Lambda}}(s) + \mathbf{K}\mathbf{\Lambda}(s) = \mathbf{P}(s), s \in [0, T] \\
&\mathbf{\Lambda}(0) = \mathbf{0}, \dot{\mathbf{\Lambda}}(0) = \mathbf{0}
\end{aligned}
\tag{13}
$$

where $\mathbf{P}(\mathbf{s})$ is given as

$$
\mathbf{P}(s) = -C_p(1 + pf - p\varphi)\mathrm{e}^{pf}(\tfrac{\partial f}{\partial \mathbf{u}})^T\Big|_{t=T-s}
\tag{14}
$$

When $\mathbf{\Lambda}(s)$ is obtained by numerical methods, $\lambda(t)$ can be obtained by $\lambda(t) = \mathbf{\Lambda}(T - t)$.

In particular, when the displacement response is considered, $f = \mathbf{L}^T\mathbf{u}$, $\partial f/\partial \mathbf{u} = \mathbf{L}^T$, $\partial f/\partial \eta_e = 0$, where $\mathbf{L}$ is a unit length vector with zeros at all degrees of freedom except at the point where the displacement is considered.

When $\mathbf{u}(\mathbf{t})$,$\dot{\mathbf{u}}(t)$,$\ddot{\mathbf{u}}(t)$ and $\lambda(t)$ have all been obtained, the objective function and the sensitivities can be evaluated by (2) (12) and the trapezoidal summation[7]

$$
\int_0^T h(t)dt \approx \sum_{n=0}^{N_s} h(t_n)w_n
\tag{15}
$$

where N_s is the number of steps used to discrete the time domain, t_n are time points and the weights are

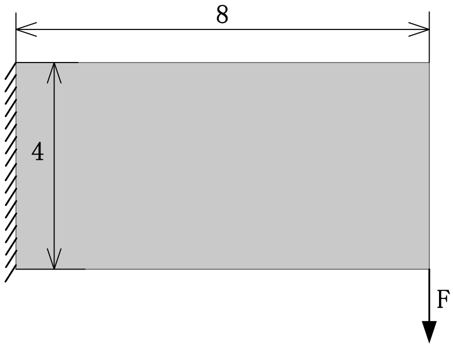
$$
2w_0 = w_1 = \cdots = w_{N_s-1} = 2w_{N_s} = T/N_s
\tag{16}
$$

6. Numerical Examples

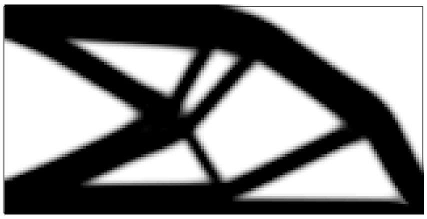
This section presents two numerical examples which minimize the maximum displacement of the loading point of the structure. In both examples, the International System of Units(SI) is adopted and Young's modulus $E_0 = 2.0E11$, Poisson's ration $\nu = 0.3$ and structural density $\rho_0 = 7800$ for fully solid material is assumed. The Rayleigh damping $\mathbf{C} = \alpha_\mathbf{r}\mathbf{M} + \beta_\mathbf{r}\mathbf{K}$ is assumed and the coefficients α_r and β_r are considered to be design-independent and are chosen as $\alpha_r = 10$ and $\beta_r = 10^{-5}$. The sensitivity filter[8] is employed to avoid the checkerboards and mesh-dependencies phenomena. The radius of the sensitivity filter is set to be three times the size of the elements used to discrete the design domain. The standard settings are used for the MMA optimizer[9]. The Newmark method is used to solve the dynamics equilibrium equation. The value of the parameter in the functional is set to be $5/f_{max}$, where f_{max} is the maximum value of the displacement during the loading phase. For comparison, a corresponding static design problem is also set up for each example, where the magnitude of the static load is the same as the maximum value of the dynamic load over loading phase.

6.1. Cantilever Beam under Sinusoidal Loading

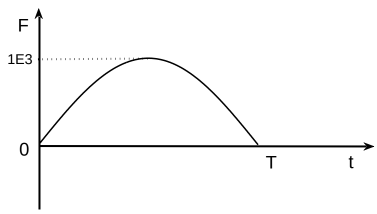
This example uses a cantilever beam design problem as shown in Fig. 1(a) to demonstrate the effectiveness of the proposed method. The design domain is a 8×4 rectangular area with thickness 0.01. A dynamic load is vertically applied at the bottom right of the structure. The prescribed volume fraction of material is set to 0.5. The dynamic loading is assumed as the sinusoidal function, and two different integration time $T = 0.02, 0.2$ are considered.



(a) Design domain and boundary condition



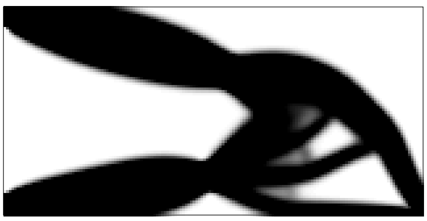
(b) Static design



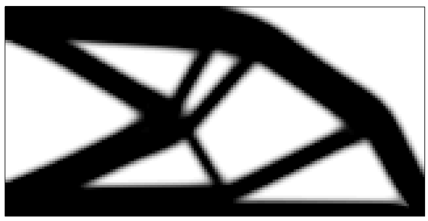
(c) Sinusoidal Load

Figure 1: Topology optimization problem of cantilever beam under sinusoidal loading

In order to solve the topology optimization problem, the design domain is discretized by square bilinear plane stress elements whose size is 0.05. The corresponding static design is shown in Fig. 1(b). The optimal designs under dynamic loading are shown in Fig. 2. They clearly show that the change rate of the dynamic load have great influence on the design result. When the load changes slowly, the dynamic design is similar to the static design. When the load changes rapidly, the dynamic design is obviously different from its static counterpart. The data listed in Table 1 show that, the dynamic design is better than the static design when the time-dependent characteristic of the load and the inertia effect are considered.



(a) $T = 0.02$



(b) $T = 0.2$

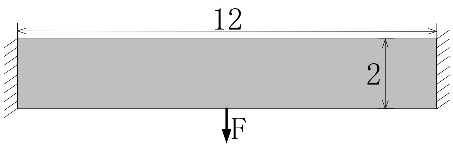
Figure 2: Topology optimization results of cantilever beam under sinusoidal loading

Table 1: Maximum displacement of the design for cantilever beam under sinusoidal loading

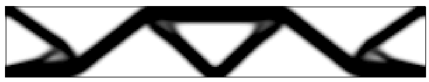
Integration time	Dynamic design	Static design
$T = 0.02$	$4.37e-5$	$5.35e-5$
$T = 0.2$	$3.72e-5$	$3.73e-5$

6.2. Clamped Beam under Cosine Loading

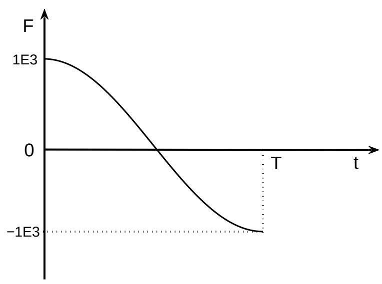
The topology optimization problem is shown in Fig. 3(a). The design domain is a 12×2 clamped rectangular area with thickness 0.01. A dynamic load is vertically applied at the centerline of the bottom of the structure. The prescribed volume fraction of material is set to 0.4. The dynamic loading is assumed as the cosine function, and four different integration time $T = 0.02, 0.05, 0.1$ and 2.0 are considered.



(a) Design domain and boundary condition



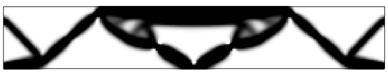
(b) Static design



(c) Cosine Load

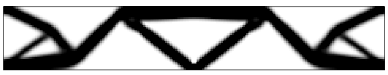
Figure 3: Topology optimization problem of clamped beam under cosine loading

In order to solve the topology optimization problem, the design domain is discretized by square bilinear plane stress elements whose size is 0.05. The corresponding static design is shown in Fig. 3(b). The optimal designs under dynamic loading are shown in Fig. 4. They clearly show that the change rate of the dynamic load have great influence on the design result. When the load changes slowly, the dynamic design is similar to the static design. When the load changes rapidly, the dynamic designs are obviously different from their static counterpart. The dynamic design corresponding to different integration time are also different from each other. The data listed in Table 2 show that, the dynamic design are better than their static counterpart when the structure subjected to the given dynamic loading. This example again demonstrate the effectiveness of the proposed method.

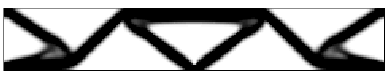


(a) $T = 0.02$

(b) $T = 0.05$



(c) $T = 0.1$



(d) $T = 0.2$

Figure 4: Topology optimization results of clamped beam under cosine loading

7. Acknowledgements

Table 2: Maximum displacement of the design for clamped beam under cosine loading

Integration time	Dynamic design	Static design
$T = 0.02$	2.20e-5	3.14e-5
$T = 0.05$	3.02e-5	4.02e-5
$T = 0.1$	3.39e-5	4.21e-5
$T = 2.0$	4.08e-5	4.25e-5

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8. References

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