Vectors for Zero Robotics students

Zero Robotics Australia

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1 Assumed Knowledge

The robots used for the Zero Robotics competition (SPHERES) were designed for NASA researchers, and are able to do some quite complex things. Although an average high school student is able to control their basic behaviour, they have many complex functions that can only be used after studying higher mathematics.

You don't have to know how to use all of these functions to be part of Zero Robotics, but the more you learn, the more interesting things you'll be able to do.

This tutorial takes you from year 7 vectors, to year 12 and beyond very quickly. We recommend you read at least up to your approximate year level as revision. Here's a chart to give you an idea of what's coming.

If you already know about...	or you've finished year...	You'll recognise the maths up to page...
The Cartesian Plane	7	4
Pythagoras' Theorem	8	5
Trigonometry	9	6

We don't expect any students to be able to quickly understand everything in this, you will need to do your own research, but it's a good starting point. If you have any questions, please contact us at zero-robotics.admin@sydney.edu.au.

Good luck, we hope you find it interesting.

2 Introduction to vectors

2.1 What are Scalars

In mathematics there are two types of numbers scalars and vectors.
A scalar is a number it has no direction.

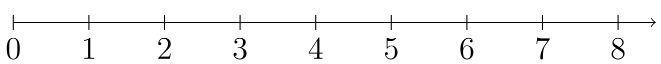


Figure 1: Number line. These numbers are scalars

A number line is made up of scalar numbers of different magnitudes. A magnitude is how large a number is.

Scalar Example: Both Ben and Jeremy are standing 1 meter away from the tree. We don't know whether they are in front, behind or to the right or left of the tree, so 1 meter is a scalar.

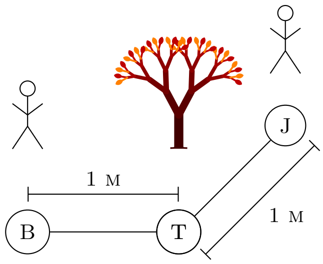
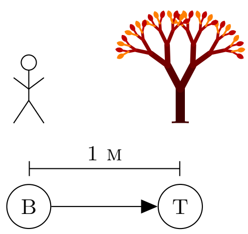


Figure 2: Ben and Jeremy are both one meter away from the tree

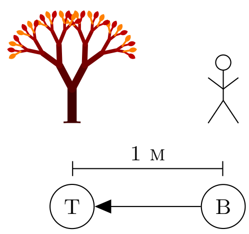
2.2 What are Vectors

A vector has a magnitude as well as direction. Using vectors, we could say Ben is 1m to the left of the tree or, that the tree in 1m to the right of Ben[1]. Now we have more information about Ben's location.

Mathematically we describe the vector from Ben to the tree, as BenTree or $\vec{BT}$ for short.



(a) Right Hand Vector example



(b) Left Hand Vector example

[1]We just said the same thing in two different ways. This is important later on.

2.3 Cartesian Plane

Now we know that a vector has a magnitude and a direction, we need a way to describe them mathematically.

To do this, we use the Cartesian Plane[2]. This allows us to describe directions as both left and right, and up and down. We call these directions, the x and the y axis.

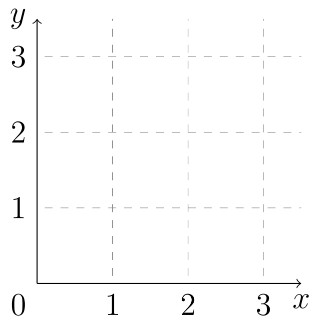


Figure 4: The Cartesian Plane

REVIEW:The Cartesian Plane

The Cartesian Plane is a 2D grid that allows you to specify the exact location of points, using an x coordinate and a y coordinate. Points are expressed in the form (x,y).

If we want to describe where Ben is, compared to the tree (going from the tree to Ben, so we write $\vec{TB}$), on a Cartesian Plane we can say that he's 1 meter in the y direction, and 2 meters in the x direction.

[2] If you haven't covered this in class yet, ask your maths teacher for help or look it up

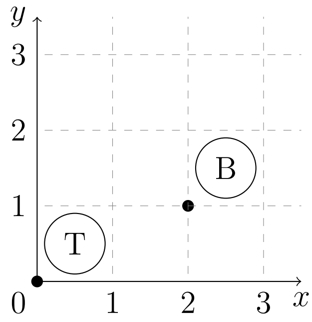


Figure 5: Tree (T) and Ben (B) on the Cartesian Plane

Mathematically, we can write this in a few different ways.

Vector notation: $\vec{TB} = [2, 1]$. This shows that it moves two metres in the x direction and one in y. We are too lazy to write in the x and y (but we know that they're there). Be careful not to mix this up with coordinates! In this tutorial, coordinates are surrounded by normal brackets ( ) and vectors have square brackets [ ].
Algebraic notation: $\vec{TB} = 2\vec{x} + 1\vec{y}$. This makes the difference between the x and y directions obvious.

CODE: Vectors in C

Vectors are represented in C as an array of the components. An example for the vector $\vec{u} = [1, 2]$ is given below:

```
float vec_u[2] = {1.0,2.0};
```

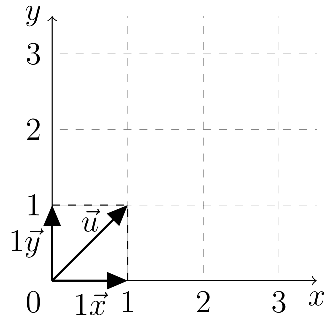
2.3.1 Starting location

In the last section the tree was at the origin (0,0), so that was the start point of the vector. This isn't always the case, but it doesn't change how we describe the vector mathematically, just how we draw it.

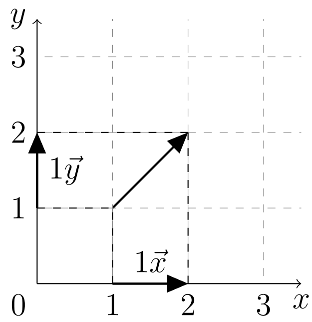
Example: you want to find the vector between object A and B, located at (1,1) and (2,2) on a Cartesian Plane. See figure 6.

You can find this vector by drawing a line between the two points. By counting the squares between them we see that the vector has gone across one square and up one square, so it can be expressed as $\vec{Vector} = 1\vec{x} + 1\vec{y}$.

There is not mathematical different between this, and a vector that starts at the origin.



(a) Origin vector



(b) Floating vector

Figure 6: Vector Examples

2.4 Finding the length of a vector

Often, we want to know how long the vector is (called its magnitude). This can be calculated using Pythagoras' Theorem[3].

REVIEW: Pythagoras' Theorum

Pythagoras' theorem says that you can find the length of the hypotenuse of a triangle, by squaring the other two sizes, then adding them. Or, mathematically: $a^2 + b^2 = c^2$

Lets look at one of the old pictures again.

[3]If you haven't covered this in class yet, ask your maths teacher for help with this section

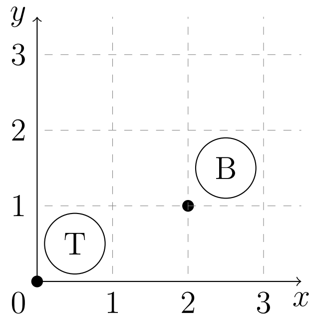


Figure 7: Tree (T) and Ben (B) on the Cartesian Plane

If we draw the vector components in it looks like a right handed triangle, so we can use Pythagoras' theorem on it. So if we want to find the distance from the Tree to Ben, we can calculate it: $|\vec{TB}| = \sqrt{2^2 + 1^2} = \sqrt{5} \approx 2.2$.

2.5 Finding where a vector is pointing

Sometimes we want to know where a vector is pointing.

This is useful when navigating along a bearing, finding how high something is, or working out how to move your SPHERE to a game item.

For some vectors, it's obvious what the angles are (figure 8), but sometimes it's not.

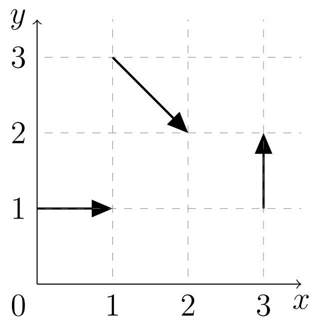
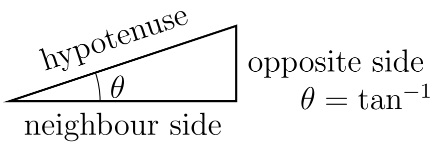


Figure 8: Examples of Obvious Vectors

Mathematically, we can use a special function in maths called inverse tan ($\tan^{-1}()$). To use this function, divide the length of the side opposite the angle you are looking for, by the length of the side next to (neighbouring) the angle you are looking for, but not the hypotenuse[4]; See figure 9.

REVIEW: Trigonometry

Trigonometry is the study of triangles. Each triangle has three sides, and three angles. If you know the magnitude of three of these values, you can determine all of the others.



$$\theta = \tan^{-1}\left(\frac{\text{opposite side}}{\text{neighbour side}}\right)$$

Figure 9: Inverse Tan Application

[4] If you haven't done trigonometry, ask a teacher for help in this section

3 Basic Vector Operations

3.1 Vector Addition

Addition can be done in two ways mathematically and geometrically.

Mathematically

Mathematically, we take the $\vec{x}$ and $\vec{y}$ components of each vector and add them together separately. For example, if we have two vectors:

$$\vec{Vector1} = 2\vec{x} + 1\vec{y}$$
$$\vec{Vector2} = 1\vec{x} - 2\vec{y}$$

the total becomes

$$\vec{Vector3} = 3\vec{x} - 1\vec{y}$$

Geometrically

To add vectors geometrically we draw them next to each other, "head-to-toe", then draw a line from the start of the first vector to the end of the last one to find our solution.

For example, if we have three vectors, $\vec{u}$, $\vec{v}$ and $\vec{w}$;

$$\vec{u} = 2\vec{x} + 3\vec{y}$$
$$\vec{v} = 3\vec{x} + 4\vec{y}$$
$$\vec{w} = 5\vec{x} + 1\vec{y}$$

we can draw them out on a Cartesian plane.

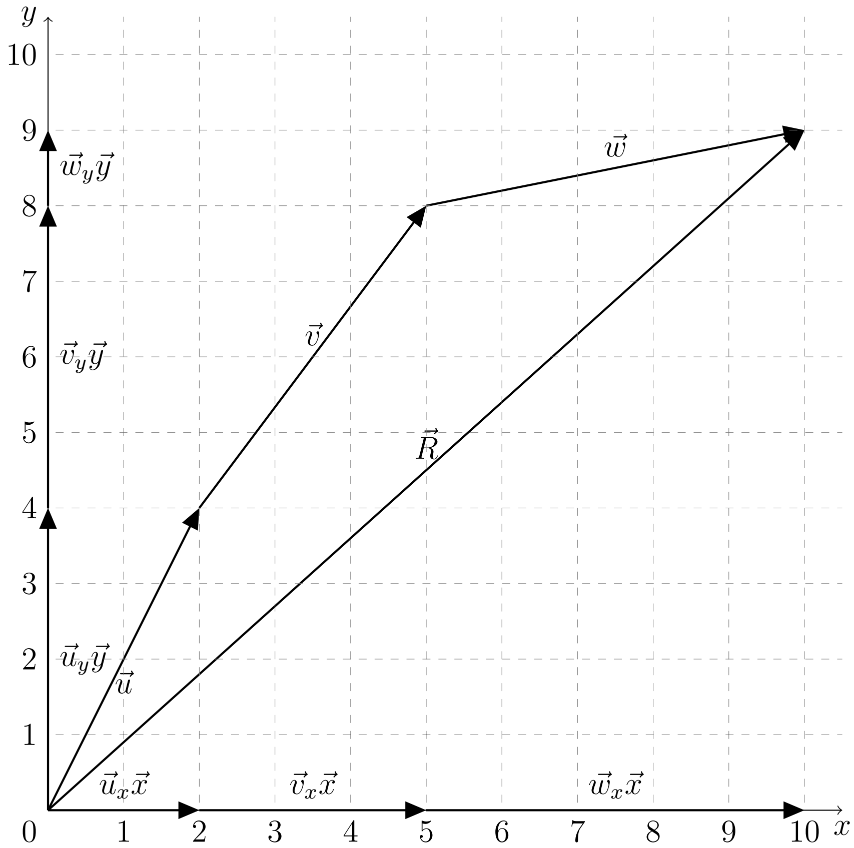


Figure 10: Vector Addition Example[5]

Then we can draw a solution vector from the start of the first vector to the end of the last vector.
The resultant vector, $\vec{R}$ has the value $\vec{R} = 10\vec{x} + 9\vec{y}$.

The geometric method makes vector addition easier to understand, but it takes longer, so we use the mathematical method.

CODE: Adding Vectors in C

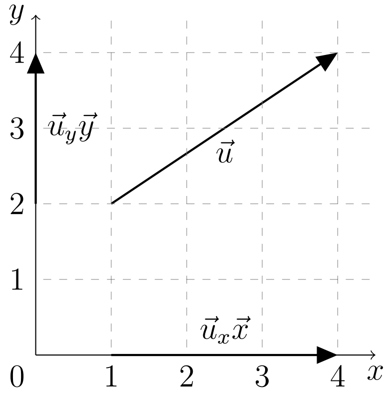
Vectors can be added together in Zero C using the function mathVecAdd(), note the last parameter (as per the example) is the number of elements in the array or the number of dimensions the vector has, **it is not the magnitude of the vector**. An example for adding $\vec{u} = [1, 2]$ and $\vec{v} = [3, 4]$ to be $\vec{R}$ is given below:

```c
float vec_u[2] = {1.0,2.0};
float vec_v[2] = {3.0,4.0};
float vec_R[2];
mathVecAdd(R, vec_u, vec_v, 2);
```

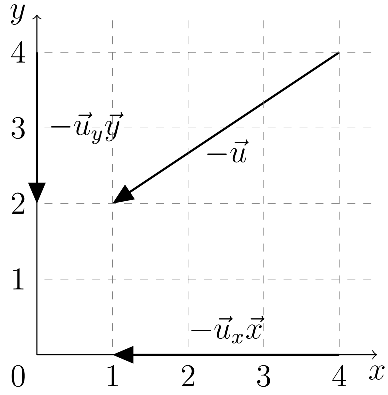
3.2 Negating Vectors

A flipped vector is a vector with each of its components multiplied by (-1).

e.g. the vector $\vec{u} = [3, 2]$ has the flipped or negated vector $-\vec{u} = [-3, -2]$.



(a) Original Vector



(b) Negated Vector

Figure 11: Vector Flipping

As you can see by the pictures, negating a vector will only change the direction of the vector, it doesn't change its magnitude. The examples shown in figure 11 show that the direction change is always 180°.

Finding the negative of vectors is useful during subtraction.

3.3 Vector Subtraction

Subtracting one vector from another is a bit trickier than subtracting one number from another.

To subtract two vectors, the vector being subtracted must be negated and then added to the other vector. Subtracting with vectors is the same as adding negative numbers to do subtraction,

e.g. $3 - 2$
$= 3 + (-2)$
$= 1.$

Vector subtraction example: Starting with vectors $\vec{u} = [5, 3]$ and $\vec{v} = [3, 1]$ find the resultant vector, $\vec{R} = \vec{u} - \vec{v}$.

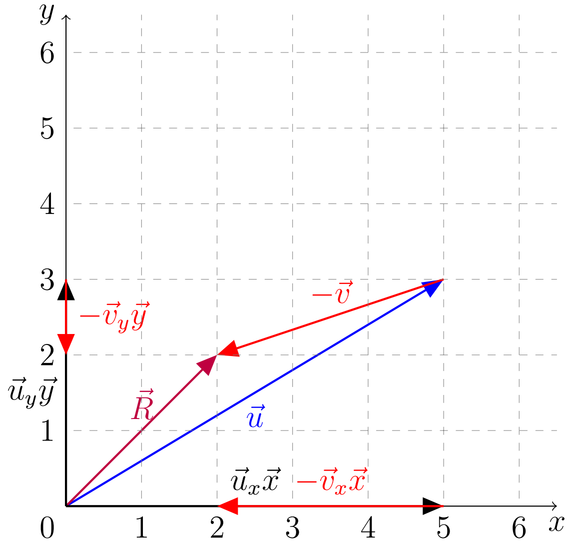


Figure 12: Vector Subtraction Example

CODE: Subtracting Vectors in C

Vectors can be subtracted from one another in Zero C using the function mathVecSubtract(), It behaves the same way as the vector addition function. An example is to take $\vec{u} = [1, 2]$ and subtract $\vec{v} = [3, 4]$ to be $\vec{R}$ is given below:

```
float vec_u[2] = {1.0,2.0};
float vec_v[2] = {3.0,4.0};
float vec_R[2];
mathVecSubtract(R, vec_u, vec_v, 2);
```

MIT Tutorial: Combining Vectors

The MIT website has a fantastic tutorial called Combining Vectors, which introduces vector addition and subtraction. Please read it before continuing.

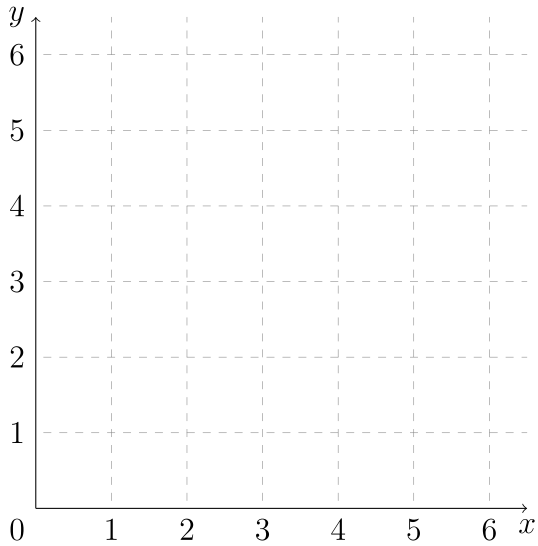
3.4 Exercises

3.4.1 Exercise 1

Mathematically solve $\vec{R} = \vec{u} + \vec{v} - \vec{w} + \vec{M} - \vec{F}$. Where $\vec{u} = -1\vec{x} + 2\vec{y}$, $\vec{v} = -3\vec{x} + -9\vec{y}$, $\vec{w} = 3\vec{x} + 8\vec{y}$, $\vec{M} = 7\vec{x} + 9\vec{y}$ and $\vec{F} = 9\vec{x} + 30\vec{y}$.

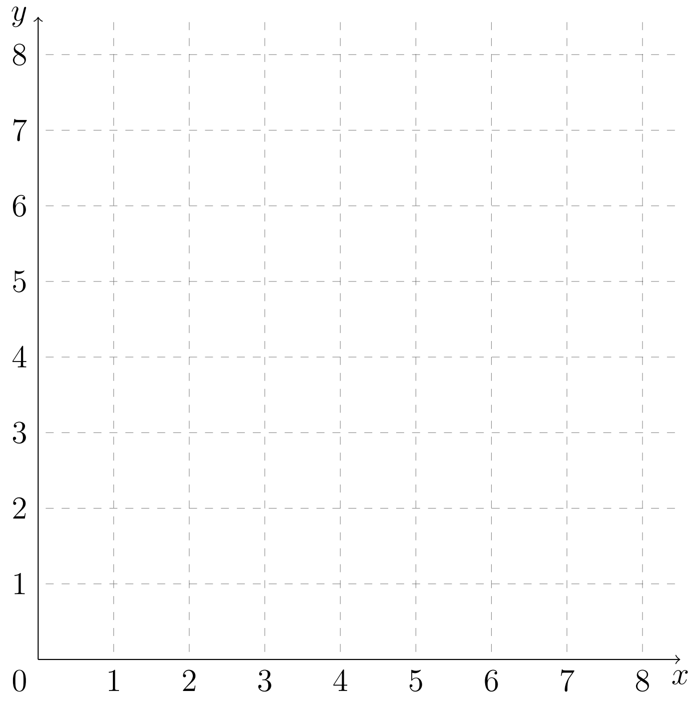
3.4.2 Exercise 2

Geometrically find $\vec{R} = \vec{u} + \vec{v} - \vec{w}$ Where $\vec{u} = [3, 5]$, $\vec{v} = [1, -3]$ and $\vec{w} = [2, -3]$. The Cartesian plane below might be useful.



3.4.3 Exercise 3

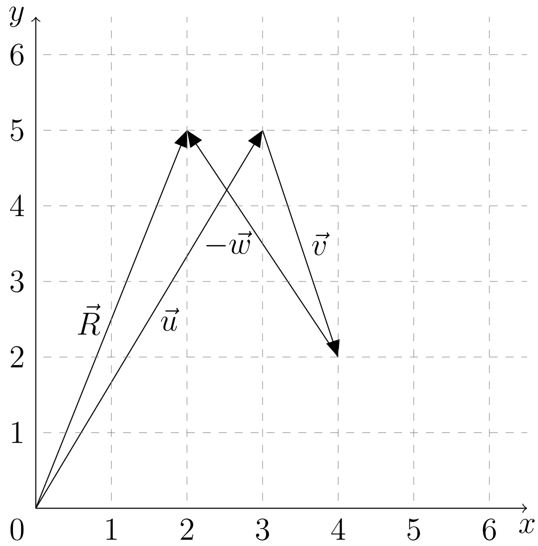
Geometrically find $\vec{R} = \vec{S} + \vec{T} + \vec{u} + \vec{v} + \vec{w}$ Where $\vec{S} = [4, 1]$, $\vec{T} = [4, 0]$, $\vec{u} = [-2, 5]$, $\vec{v} = [5, 2]$ and $\vec{w} = [0, -7]$. The Cartesian plane below might be useful.



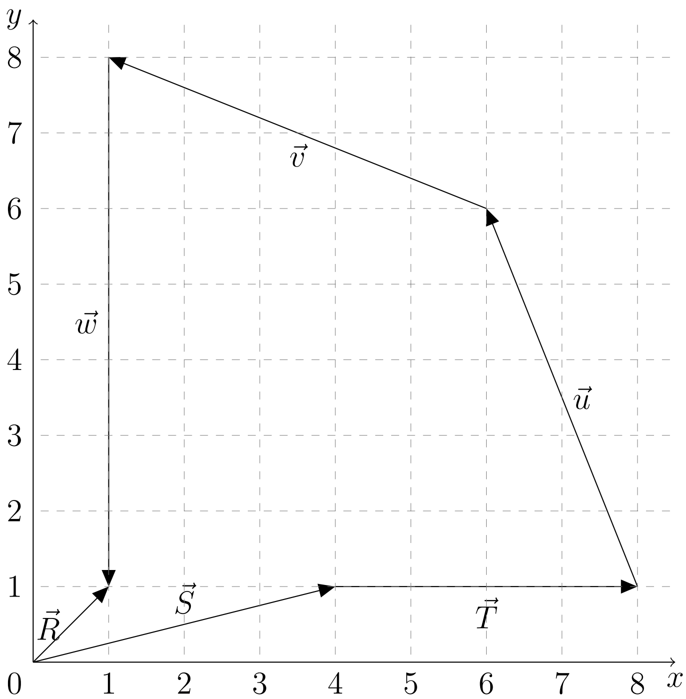
3.5 Solutions

Exercise 1 Sample Solution

$$\begin{aligned}\vec{R} &= \vec{u} + \vec{v} - \vec{w} + \vec{M} - \vec{F} \\ &= (-1\vec{x} + 2\vec{y}) + (-3\vec{x} + -9\vec{y}) - (3\vec{x} + 8\vec{y}) + (7\vec{x} + 9\vec{y}) - (9\vec{x} + 30\vec{y}) \\ &= -9\vec{x} - 36\vec{y}\end{aligned}$$



Exercise 2 Sample Solution; $\vec{R} = [2, 5]$



Exercise 3 Sample Solution; $\vec{R} = [1, 1]$

4 Intermediate Vector Operations

4.1 Vector Multiplication

Just like scalar numbers, vectors can be multiplied together. Unlike scalars, there are two different types of vector multiplication; **dot products**, and **cross products**. Dot products take two vectors, and return a scalar number, whereas cross products take two vectors and return another, new vector. Cross products can only be used in three dimensional space (i.e. where there is an $\vec{x}$, $\vec{y}$ and $\vec{z}$ axis).

4.2 Dot Product

4.2.1 Mathematical Definition

Mathematically the common components are multiplied with one another and then summed together.
The formula is; $\vec{R} = \vec{u} \cdot \vec{v} = \vec{u}_x \times \vec{v}_x + \vec{u}_y \times \vec{v}_y$.

Example: For two vectors; $\vec{u} = 2\vec{x} + 4\vec{y}$, and $\vec{v} = 4\vec{x} + 3\vec{y}$
the dot product is: $\vec{u} \cdot \vec{v} = 2 \times 5 + 4 \times 3 = 22$.

Dot products are also commutative, which means that it doesn't matter which order the terms go, the result is the same.
Example: If we reverse the order of the two vectors in the last example, the answer is the same. $\vec{R} = \vec{v} \cdot \vec{u} = \vec{u} \cdot \vec{v} = 22$

4.2.2 Geometric Definition

Geometrically, the dot product is the projection of one vector along the other, multiplied.[6]

The formula is: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$ where $|\vec{u}|$ is the magnitude of $\vec{u}$ and θ is the angle between the vectors.

If we break it down we can see that $\vec{u} \cdot \vec{v} = |\vec{u}| \left(|\vec{v}|\cos(\theta)\right)$ where $\left(|\vec{v}|\cos(\theta)\right)$ is the amount of $\vec{v}$ that goes in the $\vec{u}$ direction.

[6] If you haven't done trigonometry yet, ask a teach for help with this section

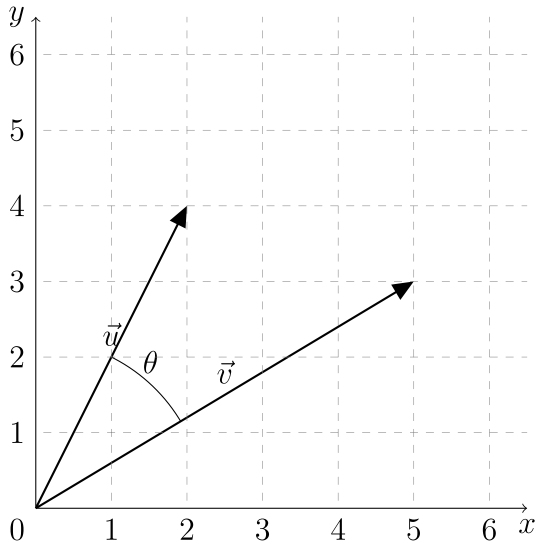


Figure 13: Vector Dot Product

CODE: Dot Product of Vectors in C

Vectors to be dot product together in Zero C using the function mathVecInner(), It behaves the same way as the vector addition and subtraction function. An example is to take $\vec{u} = [1, 2]$ and dot product $\vec{v} = [3, 4]$ with to get $\vec{R}$:

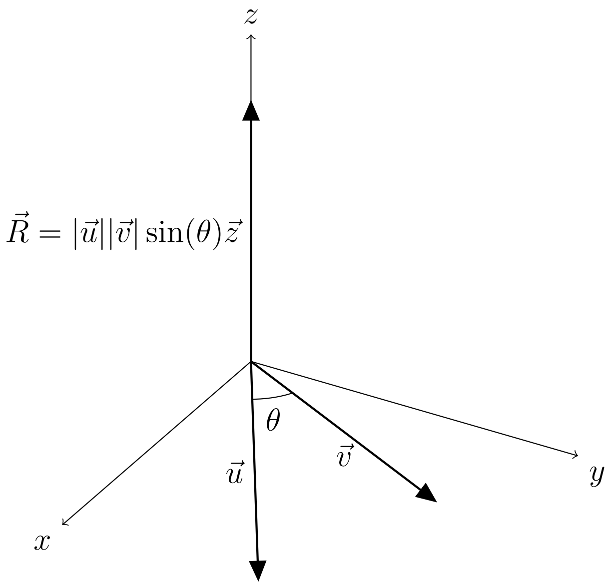
```
float vec_u[2] = {1.0,2.0};
float vec_v[2] = {3.0,4.0};
float R;
mathVecInner(R, vec_u, vec_v, 2);
```

4.3 Cross Product

A cross product is a different sort of vector multiplication that can only be used in 3D space.

It takes two vectors, and returns a new vector which is perpendicular to both original vectors. The new vector has a magnitude equal to the parallelogram sketched out by the original vectors.

4.3.1 Geometric Description



4.3.2 Mathematically

The cross product is defined mathematically as $R = \vec{u} \times \vec{v} = (\vec{u}_y\vec{v}_z - \vec{u}_z\vec{v}_y)\vec{x} - (\vec{u}_x\vec{v}_z - \vec{u}_z\vec{v}_x)\vec{y} + (\vec{u}_x\vec{v}_y - \vec{u}_y\vec{v}_x)\vec{z}$.

In Zero Robotics, these can be used to find the best ...

4.3.3 Engineering Uses

Engineering applications are far more abundant as they are used to describe systems of levers. An example with a seesaw is given below.

The cross product may also be defined as $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}|\sin(\theta)\vec{n}$, where $\vec{n}$ is the vector normal to the plane formed by the other two vectors and θ is the angle between them.

To balance the see saw the lever action or moment, force$\times$distance must be equal to one another. $\vec{u} = -2\vec{y}$ N, $\vec{v} = -\vec{v}_y\vec{y}$ N, $\vec{PA} = -5\vec{x}$ m and $\vec{PB} = 3\vec{x}$ m.

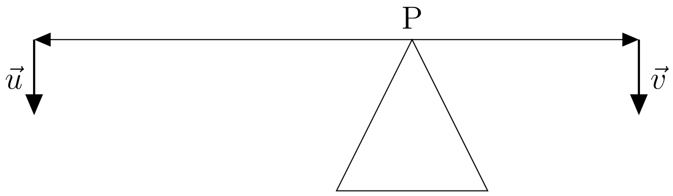


Figure 14: Cross Product Example

Thus the balance statement is:

$$\begin{aligned}
\vec{u} \times \vec{PA} + \vec{v} \times \vec{PB} &= 0 \\
\vec{v} \times \vec{PB} &= -\vec{u} \times \vec{PA} \\
-(-\vec{v}_y \times 3)\vec{z} &= (-2 \times -5)\vec{z} \\
3\vec{v}_y &= 10 \\
\vec{v}_y &= \frac{10}{3}\text{N} \\
\therefore \vec{v} &= \frac{10}{3}\vec{y}\ \text{N}
\end{aligned}$$

CODE: Cross Product of Vectors in C

An example is to take $\vec{u} = [1, 2]$, $\vec{v} = [3, 4]$; $\vec{R} = \vec{u} \times \vec{v}$ is given below:

```
float vec_u[2] = {1.0,2.0};
float vec_v[2] = {3.0,4.0};
float vec_R[3];
mathVecCross(R, vec_u, vec_v);
```

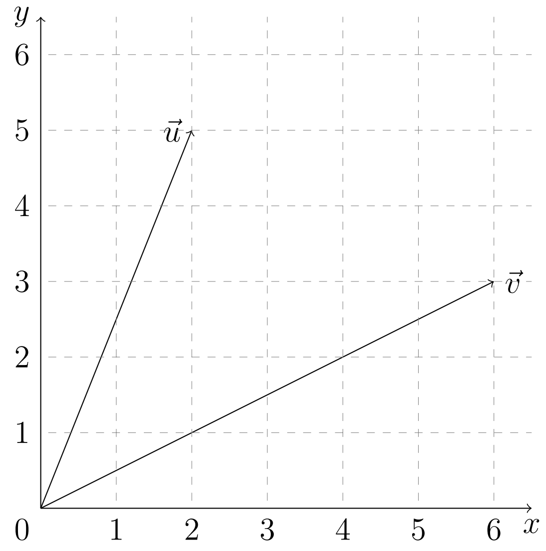
4.4 Exercises

4.4.1 Exercise 1

Mathematically solve $\vec{R}_1 = \vec{u} \cdot \vec{v}$; $\vec{R}_2 = \vec{v} \cdot \vec{u}$; $\vec{R}_3 = \vec{u} \times \vec{v}$ and $\vec{R}_4 = \vec{v} \times \vec{u}$. Where $\vec{u} = -1\vec{x} + 2\vec{y}$ and $\vec{v} = -3\vec{x} + -9\vec{y}$.

4.4.2 Exercise 2

Geometrically find the dot product and cross product from the diagram below of $\vec{u}$, $\vec{v}$.



4.4.3 Exercise 3

Find $\vec{R} = (\vec{u} \times \vec{v}) \cdot \vec{w}$; where $\vec{u} = [3, 7]$; $\vec{v} = [5, 2]$ and $\vec{w} = [1, 1]$

4.5 Solutions

Exercise 1 Solutions

$\vec{R}_1 = -15 \quad \vec{R}_2 = -15$

$\vec{R}_3 = 15\vec{z} \quad \vec{R}_4 = -15\vec{z}$

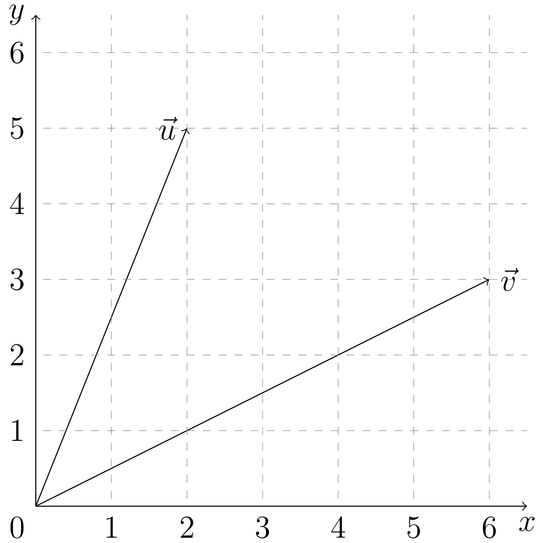


Figure 15: Exercise 2 Sample Solution

$|\vec{u}| = \sqrt{29}$

$|\vec{v}| = 3\sqrt{5}$

$\theta = 41.6°$

Thus the dot product is 9.00 and the cross product is $7.99\vec{z}$.

Exercise 3 Solutions

$\vec{R} = [0, 0] = 0$