11th World Congress on Structural and Multidisciplinary Optimisation
07th -12th, June 2015, Sydney Australia

A Novel Anti-optimization Method for Structural Robust Design under Uncertain Loads

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1. Abstract
Robust design aims to find a configuration set in which the structural performance is least sensitive to uncertain parameters. The robust design method under loads uncertainty is a nested optimization. In each iteration process, the worst-case constraints can be achieved by the anti-optimization, and then the outer loop performs the optimization under the constraints. In this paper, a new anti-optimization technique is used to alleviate the computational burden with the help of the axial stress forces in members of the truss under each load respectively. The axial forces are the main information in the anti-optimization process which can be easily found by finite element analysis, and the sign of the forces decide the value of the loads in the corresponding constraints. The optimum of the structure is achieved by minimizing the value of the objective function subject to the worst-case constraints under uncertainty loads while the robustness is ensured by the anti-optimization process conducting the worst-case-scenario. The robust design of a 10-bar and a 25-bar truss under uncertain loads are carried out to demonstrate the effectiveness of the present method.
2. Keywords: robust design; uncertain loads; worst-case; anti-optimization;

3. Introduction
The optimization of structure is always performed under deterministic loading conditions. However, the loads are uncertain in most practical situations, and the designer must take the effects of the uncertainty into consideration. The existing popular approaches to describe uncertainties can be classified as probabilistic methods [1,2] and the convex model methods [3,4,5,]. The probabilistic methods require a detailed description of the uncertainties, which can hardly be used in practical engineering. In contrast, the convex model methods, which just require the knowledge of the bounds of the uncertain parameters, so it is well suited for cases where lack of information makes the implementation of probabilistic approaches very difficult.
The theory of convex models was investigated in detail by Ben-Haim and Elishako [6]and Elishako et al. [7], and it is implemented on the constraints by the use of an anti-optimization process, which finds the worst-case produced by the uncertain load condition. The anti-optimization is actually a two-level optimization problem. At the outer level, it obtains the best design by optimizing the design variables, while at the inner level it seeks the worst condition for a given design from anti-optimization.
After the anti-optimization method is proposed, many researchers have applied it to the theory and practice. Marco Lombardi et al.[8] described a technique for design under uncertainty based on the worst-case-scenario technique of anti-optimization, the method can alleviated the computational burden. Marco Lombardi [9] compared two different approaches which use anti-optimization, namely a nested optimization and a two steps optimization, where anti-optimization is solved once for all constraints before starting the optimization. Stewart McWilliam [10] present two new methods for solving constraint equations by anti-optimization method of uncertain structures using interval analysis. Zhiping Qiu [11] studied the anti-optimization problem of structures with uncertain design variables by combing the conventional optimization and interval analysis.
In this paper, a new anti-optimization technique is used to alleviate the computational burden with the help of the axial stress forces in members of the truss under each load respectively. The axial forces are the main information in the anti-optimization process which can be easily found by finite element analysis, and the signs of the forces decide the value of the loads in the corresponding constraints.

4. Problem formulation
In the deterministic formulation of structural optimization problems, the design variables and parameters are assumed deterministic and the objective function as well as the constraints is referred to their nominal values. The classical formulation of structural optimization problem can be mathematically expressed as:

$$\begin{aligned} &\text{find} && \mathbf{X} \\ &\text{minimize} && f(\mathbf{X}) \\ &\text{subject to} && g_i(\mathbf{X}) \le 0 \ (i = 1, 2, ..., k) \end{aligned} \tag{1}$$

Where $\mathbf{X} = (x_1, \ldots, x_n)^T$ is the n-dimensional vector of design variables, $f(\mathbf{X})$ the objective function, $g_i(\mathbf{X})$ the inequality constraint functions, and k is the total number of constraints. For simplicity, the equality constraint functions are not described here.

However, one may observe that the objective function $f(\mathbf{X})$ and/or the constraints $g_i(\mathbf{X})$ may depend on some design parameters subject to uncertainty. In this paper, only loads uncertainty is concerned, and the stress and displacement are the constraints, and interval variables are referred to the loads that vary about their nominal values and the design variable are controllable design parameters that need to be determined by the designer, such as the area of truss elements. Considering for the sake of simplicity the case where the uncertainty is limited to the constraint functions $g_i(\mathbf{X})$, so Eq. (1) becomes:

$$\begin{aligned}
&\text{find} && \mathbf{X} \\
&\text{minimize} && f(\mathbf{X}) \\
&\text{subject to} && \underline{\sigma}_i^{allow} \le \sigma_i(\mathbf{X},\mathbf{P}) \le \overline{\sigma}_i^{allow} \; (i = 1,\ 2, \ldots,\ k) \\
& && \underline{u}_j^{allow} \le u_j(\mathbf{X},\mathbf{P}) \le \overline{u}_j^{allow} \; (j = 1,\ 2, \ldots,\ t) \\
& && \mathbf{P}^L \le \mathbf{P} \le \mathbf{P}^U
\end{aligned} \tag{2}$$

in where it is assumed that $\mathbf{P} = [\mathrm{P}_1, \mathrm{P}_2, \ldots, \mathrm{P}_n]$ is the loads uncertainty design parameter vector that only the bounds are known, $\mathbf{P}^{\mathrm{L}}$ and $\mathbf{P}^{\mathrm{U}}$ denote the lower and upper bound of $\mathbf{P}$, i.e. $\mathbf{P} \in \mathbf{P}^{\mathbf{I}} = [\mathbf{P}^{\mathrm{L}}, \mathbf{P}^{\mathrm{U}}]$. $\sigma_i(\mathbf{X},\mathbf{P})$ and $u_j(\mathbf{X},\mathbf{P})$ are the implicit response variables: nodal displacement and element stress. $\underline{\sigma}_i^{allow}$ and σ_i^{allow} are the allowable stress of the ith element, k is the number of element. $\underline{u}_j^{allow}$ and u_j^{allow} are the maximum allowable displacement of the jth node, t is the number of node.

In formulation (2), we require that $\sigma_i(\mathbf{X},\mathbf{P})$ and $u_j(\mathbf{X},\mathbf{P})$ are satisfied for all the possible values of $\mathbf{P}$, and the structural robust optimization problem can be expressed as an alternative formulation is:

$$\begin{aligned}
&\text{find} && \mathbf{X} \\
&\text{minimize} && f(\mathbf{X}) \\
&\text{subject to} && \max_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} \sigma_i(\mathbf{X},\mathbf{P}) - \sigma_i^{allow} \le 0 \; (i = 1,\ 2,\ \ldots,\ k) \\
& && \underline{\sigma}_i^{allow} - \min_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} \sigma_i(\mathbf{X},\mathbf{P}) \le 0 \; (i = 1,\ 2,\ \ldots,\ k) \\
& && \max_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} u_j(\mathbf{X},\mathbf{P}) - \overline{u}_j^{allow} \le 0 \; (j = 1,\ 2,\ \ldots,\ t) \\
& && \underline{u}_j^{allow} - \min_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} u_j(\mathbf{X},\mathbf{P}) \le 0 \; (j = 1,\ 2,\ \ldots,\ t) \\
& && \mathbf{P}^L \le \mathbf{P} \le \mathbf{P}^U
\end{aligned} \tag{3}$$

The maximization of $\sigma_i(\mathbf{X},\mathbf{P})$ and $u_j(\mathbf{X},\mathbf{P})$ over $\mathbf{P}^{\mathrm{I}}$ is a process of finding the worst value of $\mathbf{P}$ for each constraint, and this is an anti-optimization process.

The worst-case set of parameters is searched at each optimization iteration process, and this can be very expensive to solve. The novel anti-optimization method proposed in the paper is to alleviate the computational burden which is a main difficulty that restricts the application of structural robust design optimization. The process is described as follows:

In the anti-optimization, the stress of each element $\sigma_i(\mathbf{X},P_\gamma)$ and the displacement of each node $u_j(\mathbf{X},P_\gamma)$ are obtained under the load P_γ. When $\sigma_i(\mathbf{X},P_\gamma) \ge 0$, the force P_γ will be replaced by P_γ^{U} which is the upper bound of P_γ in the constraint $\max_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} \sigma_i(\mathbf{X},P_\gamma) - \overline{\sigma}_i^{allow} \le 0$, and the force P_γ will be replaced by P_γ^{L} which is the lower bound of P_γ in the constraint $\underline{\sigma}_i^{allow} - \min_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} \sigma_i(\mathbf{X},p_\gamma) \le 0$. Similarly, when $u_j(\mathbf{X},\mathrm{P}_\gamma) \ge 0$, the force P_γ will be replaced by P_γ^{U} which is the upper bound of P_γ in the constraint $\max_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} u_j(\mathbf{X},P_\gamma) - \overline{u}_j^{allow} \le 0$ the force P_γ will be instead with P_γ^{L} which is the lower bound of P_γ in the constraint $\underline{u}_j^{allow} - \min_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} u_j(\mathbf{X},P_\gamma) \le 0$ .In the converse, when $\sigma_i(\mathbf{X},P_\gamma) < 0$, the force P_γ will be replaced by P_γ^{L} in the constraint $\max_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} \sigma_i(\mathbf{X},P_\gamma) - \overline{\sigma}_i^{allow} \le 0$, and the force P_γ will be replaced by P_γ^{U} in the constraint $\underline{\sigma}_i^{allow} - \min_{\mathbf{P}\in\mathbf{P}^{\mathbf{I}}} \sigma_i(\mathbf{X},p_\gamma) \le 0$. Similarly, when $u_j(\mathbf{X},P_\gamma) < 0$, the force P_γ will be replaced by P_γ^{L} in the constraint $\max\limits_{\mathbf{P}\in\mathbf{P}^{\mathrm{I}}} u_j\left(\mathbf{X},P_\gamma\right)-\bar{u}_j^{allow} \leq 0$, the force P_γ will be replaced by P_γ^{U} in the constraint $\underline{u}_j^{allow} - \min\limits_{\mathbf{P}\in\mathbf{P}^{\mathrm{I}}} u_j\left(\mathbf{X},P_\gamma\right)\leq 0$. By the analysis of the anti-optimization process, the worst-case can be achieved easily. And then the robust optimization changed to a deterministic optimization problem. In order to improve the efficiency of convergence, the initial values of design parameters are inherited by the result of certainty optimization. The flow diagram of the optimization process is showed as Figure 1.

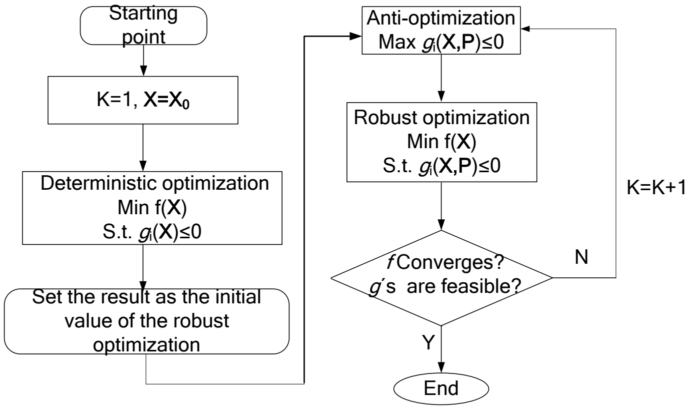

Figure 1: Optimization process

5. Numerical examples

5.1. Example 1: 10-bar aluminium truss

Consider the well-known 10-bar truss which is made of aluminum as shown in Fig. 2. The characteristics of the truss are as follows: the modulus of elasticity E is 68 948MPa(10 000ksi), the weight density ρ is 2768 kg/m^3(0.1 lb/in.^{3})and the length L of each of the vertical and horizontal bars is 9.144 m (360 in.). The maximum allowable stress in each member is the same for tension and compression. The allowable stress $\sigma_{j,allowable}$ is 172.37MPa (25ksi) for all bars except bar 9, for which the allowable stress is 517.11MPa (75ksi). The maximum allowable vertical displacement $\delta_{2,allowable}$ at joint 2 is 0.1270 m (5in.). The cross-sectional area of member j is A_j and the minimum gauge constraint of each member A_{min} is 0.645cm^2(0.1in.^{2}). The joint 4 is subjected to a vertical load P_1 while the joint 2 is subjected to both a vertical load P_2 and a horizontal load P_3 as shown in Fig. 2. Their nominal values are $\bar{P}_1$ =444.8kN (100 kip), $\bar{P}_2$ =444.8kN (100 kip), and $\bar{P}_3$ =1779.2kN (400 kip) respectively.

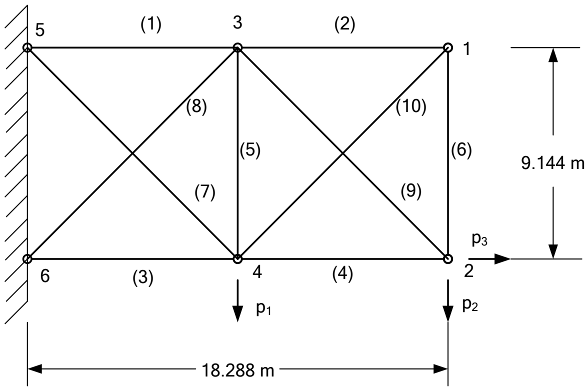

Figure 2: 10-bar truss

Following the equilibrium and compatibility equations, one may easily obtain the axial forces $N_i\,(\mathrm{i}=1,2,\ldots,10)$ in the members and the vertical displacement δ_2 at joint 2 as follows:

$N_1 = p_2 - \frac{\sqrt{2}}{2}N_8$ (4) $N_2 = -\frac{\sqrt{2}}{2}N_{10}$ (5) $N_3 = -p_1 - 2p_2 + p_3 - \frac{\sqrt{2}}{2}N_8$ (6)

$$N_4 = -p_2 + p_3 - \frac{\sqrt{2}}{2} N_{10} \quad (7) \quad N_5 = -p_2 - \frac{\sqrt{2}}{2} N_8 - \frac{\sqrt{2}}{2} N_{10} \quad (8) \quad N_6 = -\frac{\sqrt{2}}{2} N_{10} \quad (9)$$

$$N_7 = \sqrt{2}(p_1 + p_2) + N_8 \quad (10) \quad N_8 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}} \quad (11) \quad N_9 = \sqrt{2} p_2 + N_{10} \quad (12)$$

$$N_{10} = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}} \quad (13) \quad a_{11} = \left(\frac{1}{A_1} + \frac{1}{A_3} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_7} + \frac{2\sqrt{2}}{A_8}\right)\frac{L}{2E} \quad (14) \quad a_{12} = a_{21} = \frac{L}{2A_5 E} \quad (15)$$

$$a_{22} = \left(\frac{1}{A_2} + \frac{1}{A_4} + \frac{1}{A_6} + \frac{2\sqrt{2}}{A_9} + \frac{2\sqrt{2}}{A_{10}}\right)\frac{L}{2E} \quad (16) \quad b_1 = \left(\frac{p_2}{A_1} + \frac{p_1 + 2p_2 - p_3}{A_3} - \frac{p_2}{A_5} + \frac{2\sqrt{2}(p_1 + p_2)}{A_7}\right)\frac{\sqrt{2}L}{2E} \quad (17)$$

$$b_2 = \left(\frac{\sqrt{2}(p_3 - p_2)}{A_4} - \frac{\sqrt{2}p_2}{A_5} - \frac{4p_2}{A_9}\right)\frac{L}{2E} \quad (18) \quad \delta_2 = \sum_{i=1}^{10} \frac{N_i^0 N_i L_i}{A_i E} \quad (19)$$

Where the values of N_i^0 are calculated from Eqs. (4)- (13) with a substitution $p_1 = p_3 = 0$ and $p_2 = 1$.

The optimal cross-sectional areas and weight are listed in Table 1 for comparison with its deterministic counterpart. They agree well with those of Lombardi [4] and Elishakoff et al. [7], which are also shown in Table 1. Stresses of all the elements and displacement of node 2occurring at various load vertices in an optimal structure are shown in Table 2, they do not exceed the allowable stresses and/or displacement requirement, and some of them reached the allowable values in some load vertices. The anti-optimization technique is just used once in the optimization progress, so the method alleviates the computational burden.

Table 1: Optimal cross-sectional areas of the 10-bar truss under different design conditions

Member no.	Cross-section area(cm^2) Minimum weight design				
	Loads with no uncertainty		Loads with 10% uncertainty		
	Present	Ref.[4]	Present	Ref.[4]	Ref.[7]
1	26.033	26.019	28.797	28.799	28.799
2	0.645	0.645	0.645	0.645	0.645
3	26.033	26.045	44.016	44.019	44.019
4	78.060	78.064	90.584	90.587	90.589
5	24.932	24.912	27.782	27.783	27.783
6	0.645	0.645	0.645	0.645	0.645
7	72.665	72.716	79.851	79.858	79.855
8	0.645	0.645	0.645	0.645	0.645
9	17.791	17.817	29.841	29.841	29.843
10	0.912	0.912	0.645	0.645	0.645
Weight(Kg)	725.08	725.34	884.41	884.46	884.47

Table 2: Stresses and displacement of node 2 δ_2 occurring at various load vertices in an optimal structure

Loads vertices Member no.	U U U	U U L	U L U	U L L	L L U	L U L	L L U	L L L
1	1.718	1.7237	1.402	1.409	1.713	1.720	1.399	1.405
2	0.967	1.046	0.712	0.791	0.968	1.047	0.713	0.791
3	1.124	0.319	1.524	0.720	1.323	0.519	1.7237	0.919
4	1.627	1.235	1.7237	1.331	1.627	1.235	1.7237	1.331
5	-1.720	-1.711	-1.412	-1.403	-1.7237	-1.715	-1.415	-1.407
6	0.967	1.046	0.712	0.791	0.968	1.047	0.713	0.791
7	1.7237	1.721	1.569	1.566	1.568	1.565	1.413	1.410
8	-1.163	-1.555	-0.811	-1.203	-0.912	-1.303	-0.560	-0.951
9	2.289	2.287	1.875	1.873	2.289	2.287	1.875	1.873
10	-1.368	-1.479	-1.007	-1.118	-1.369	-1.480	-1.008	-1.119
δ_2	0.101	0.127	0.065	0.092	0.091	0.118	0.056	0.082

3.2. Example 2: 25-bar steel truss

The 25-bar steel truss as shown in Fig. 3 is considered here. The Young's modulus of the truss E is 199 949.2MPa (2.9e10^{4}ksi). The length L of each of the vertical and horizontal bars is 15.24 m(600 in.). The joint 12 is hinge-supported while Joints 6, 8 and 10 are roller-supported. Joints 11, 9 and 7 are subjected to vertical loads P_1, P_2 and P_3 respectively while Joint 1 is subjected to a horizontal load P_4 as shown in Fig. 3. Their nominal values are $\overline{P}_1 = \overline{P}_3 = 1779.2KN(400kip)$, $\overline{P}_2 = 2224KN(500kip)$, and $\overline{P}_4 = 1334.4KN(300kip)$ respectively. The maximum allowable stress in each member is the same whether in tension or compression. The allowable stress $\sigma_{j,allowable}$ is 172.37MPa (25ksi) for all bars. Only the stress constraints are considered in the analysis. The axial forces N_j in members can be easily found by finite element analysis.-The cross-sectional area of member j is A_j and the minimum gauge constraint of each member A_{min} is 0.645cm^2(0.1 in.^{2}).

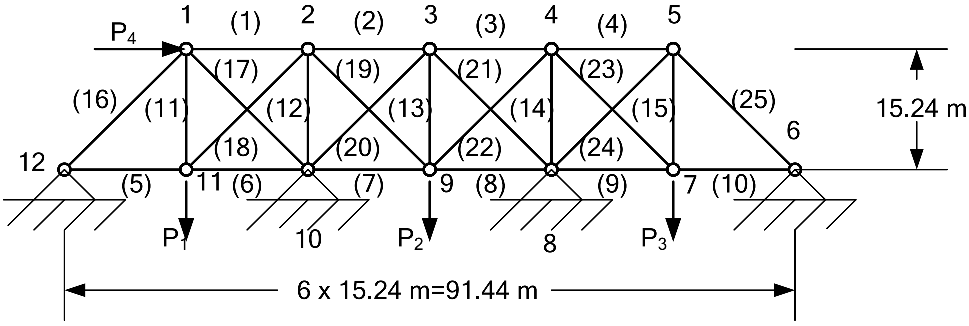

Figure 3: 25-bar truss

Similar to Example 1, this problem of minimum volume design is solved using the proposed model with the truss under nominal loadsP_1, P_2 andP_3 with 10% uncertainty and load P_4 with 20% uncertainty.
The optimal cross-sectional areas and weight are listed in Table 1 for comparison with those of no uncertainty. They agree well with those of Lombardi [4] and Ganzerli, S.et al. [12] also shown in Table 3. Stresses of all the elements occurring at various load vertices in an optimal structure are computed also, they do not exceed the allowable stresses and/or displacement requirement, and some of them reached the allowable values in some load vertices.

Table 3: Optimal cross-sectional areas of the 25-bar truss under different design conditions

Member no.	Cross-section area(cm^2)				
	Minimum weight design				
	Loads with no uncertainty		Loads P_1 through P_3 with 10% uncertainty and load P_4 with 20% uncertainty		
	Present	Ref.[4]	Present	Ref.[4]	Ref.[12]
1	19.677	19.666	27.139	27.374	26.419
2	0.645	0.645	0.645	0.645	0.645
3	1.441	1.442	6.223	6.234	5.117
4	0.645	0.645	1.633	1.574	4.193
5	74.740	74.615	90.066	89.793	89.514
6	23.575	23.536	35.879	35.555	35.687
7	0.645	0.645	0.645	0.645	0.645
8	0.645	0.645	3.868	3.896	2.740
9	0.645	0.645	0.823	0.839	2.107
10	34.860	34.648	38.913	38.640	39.669
11	52.263	52.146	65.181	65.006	65.337
12	82.674	82.708	92.387	92.593	91.671
13	64.923	64.914	71.306	71.310	71.385
14	66.725	66.778	73.878	73.884	73.760
15	68.932	68.937	75.702	75.710	75.692
16	3.9713	3.965	25.670	25.644	25.340
17	77.749	77.711	93.160	92.897	93.371
18	72.191	72.237	78.817	79.097	78.347
19	44.736	44.730	52.087	52.077	51.580
20	44.719	44.730	52.008	52.035	51.518

21	47.061	47.073	51.965	51.989	52.049
22	45.953	45.946	50.977	50.961	50.772
23	48.557	48.492	53.574	53.530	53.535
24	48.421	48.492	53.560	53.611	53.412
25	48.930	48.999	54.571	54.645	56.101
Volume(cm^3)	1.791×10^6	1.790×10^6	2.111×10^6	2.111×10^6	2.111×10^6

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