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On the numerical optimization of multi-load spatial Michell trusses using a new adaptive ground structure approach

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1. Abstract
In this paper a new method of solving large-scale linear programming problems related to Michell trusses, generalized to multiple load conditions and three-dimensional domains, is proposed. The method can be regarded as an extension of the adaptive ground structure methods developed recently by the first author. In the present version both bars and nodes can be switched between active and inactive states in subsequent iterations allowing significant reduction of the problem size. Thus, the numerical results can be attained for denser ground structures giving better approximation of exact solutions to be found. The examples of such exact solutions (new 3D Michell structures), motivated by the layouts predicted numerically, are also presented and can serve as benchmark tests for future methods of numerical optimization of structural topology in 3D space.
2. Keywords: 3D Michell structures, multiple load cases, adaptive ground structure method, linear programming, active set and interior point methods, new exact solutions for structural topology optimization.

3. Introduction
In spite of some inherent limitations (e.g. neglecting stability requirements for compression bars), the classical theory of Michell structures plays an important role in structural topology optimization, by enabling the derivation of exact analytical solutions for the least-weight trusses capable of transmitting the applied loads to the given supports within limits on stresses in tension and compression. Thus the exact solutions derived by means of this theory may serve as valuable benchmarks for any structural topology optimization method.
In general, the exact solutions are very hard to obtain since they require in advance a good prediction of the optimal layouts. Fortunately, this difficult task of predicting the optimal layout can effectively be carried out numerically using the adaptive *ground structure* method developed recently by the first author. In this method the solution is achieved iteratively using a small number of properly chosen active bars from the huge set of possible connections. The problem of enormous number of potential bars becomes particularly hard for space trusses subjected to multiple load conditions because the optimization problem grows very rapidly and becomes too large even for supercomputers.
In this paper a new and more advanced solution method for large-scale optimization problems is proposed. It is based on adaptive activating of new bars in the ground structure and eliminating large number of unnecessary bars. Moreover, in this new method the nodes can also be switched between active and inactive states. This is particularly important for 3D problems because the optimal 3D trusses tend to assume forms of lattice surfaces (shell-like structures) while most of design space becomes empty. As a result, the size of the problem can be significantly reduced. The method makes it possible to obtain new numerical solutions for deriving new optimal topologies for 3D Michell structures. A class of new 3D exact solution inspired by numerical results is also presented. The method proposed in the present paper is a natural extension of the adaptive ground structure methods developed by Gilbert and Tyas [1], Pritchard et al. [3] and Sokół [5-7].
Concluding, the aim of this paper is two-fold: a) to develop a reliable and efficient optimization method based on the adaptive ground structure approach, and b) to obtain substantially new exact solutions of spatial Michell trusses subjected to multiple load cases.

4. Primal and dual forms of the optimization problems of multi-load plastic design
Any optimization problem can be written in different forms which are mathematically equivalent but can lead to significantly different calculation times using the given optimization method (see [7]). In other words, the formulation of the optimization problem should be matched to the method applied.
According to well-known duality principles, the plastic design optimization problem can be written in either primal or in dual form. Both of them play an important role in the proposed method and should be considered together.
The most concise formulation of plastic design optimization problem for multi-load cases can be written as follows:

$$\begin{aligned}
\min_{\mathbf{A},\mathbf{S}_{(l)}\in\mathbf{R}^M} \quad & V=\mathbf{L}^T\mathbf{A} \\
s.t. \quad & \mathbf{B}^T\mathbf{S}_{(l)}=\mathbf{P}_{(l)} && \text{for all load cases} \\
& -\mathbf{A}\sigma_C \le \mathbf{S}_{(l)} \le \mathbf{A}\sigma_T && l=1,2,\ldots,K
\end{aligned} \tag{1}$$

where V is the total volume of the structural material in the truss of M potential bars; $\mathbf{L}$ is the vector of lengths of bars; $\mathbf{B}$ is the geometric matrix; vectors $\mathbf{P}_{(l)}$ define nodal forces for the given load cases l = 1, 2, …, K, where K is the number of load cases; $\mathbf{S}_{(l)}$ is the vector of member forces for load case l; $\mathbf{A}$ is the vector of cross-section areas (the main design variables); σ_T and σ_C denote the permissible stresses in tension and compression, respectively.
The primal form (1) is not convenient for direct application of *simplex* or *interior point method* and it is recommended to convert it to a more applicable form (see [7] for details). The inequalities $(1)_3$ can be converted to equality constraints using properly adjusted slack variables $\mathbf{c}_{(l)}$ and $\mathbf{t}_{(l)}$,

$$\begin{aligned}
&\sigma_T\mathbf{A}-\mathbf{S}_{(l)}-\frac{\sigma_T+\sigma_C}{\sigma_C}\mathbf{c}_{(l)}=\mathbf{0}, \quad \mathbf{c}_{(l)}\ge\mathbf{0} \\
&\sigma_C\mathbf{A}+\mathbf{S}_{(l)}-\frac{\sigma_T+\sigma_C}{\sigma_T}\mathbf{t}_{(l)}=\mathbf{0}, \quad \mathbf{t}_{(l)}\ge\mathbf{0}
\end{aligned} \tag{2}$$

which then allow elimination of original design variables $\mathbf{A}$ and $\mathbf{S}_{(l)}$

$$\mathbf{A}=\frac{\mathbf{t}_{(l)}}{\sigma_T}+\frac{\mathbf{c}_{(l)}}{\sigma_C} \quad \text{and} \quad \mathbf{S}_{(l)}=\mathbf{t}_{(l)}-\mathbf{c}_{(l)} \tag{3}$$

Note that $\mathbf{c}_{(l)}$ and $\mathbf{t}_{(l)}$ are the vectors of slack variables which can be interpreted as the additional forces which can be added without violating the restrictions of permissible stresses $(1)_3$ (i.e. they denote not forces itself but unused reserves of internal forces).
Using (3) the original problem (1) can be converted to the standard linear programming problem with the following primal form:

$$\begin{aligned}
\min_{\mathbf{t}_{(l)},\mathbf{c}_{(l)}\in\mathbf{R}^M} \quad & V=\sum_l \mathbf{L}^T\left(\frac{\mathbf{t}_{(l)}}{\sigma_T}+\frac{\mathbf{c}_{(l)}}{\sigma_C}\right) \\
s.t. \quad & \mathbf{B}^T(\mathbf{t}_{(l)}-\mathbf{c}_{(l)})=\mathbf{P}_{(l)} && \text{for } l=1:K \\
& \frac{\mathbf{t}_{(l+1)}}{\sigma_T}+\frac{\mathbf{c}_{(l+1)}}{\sigma_C}=\frac{\mathbf{t}_{(l)}}{\sigma_T}+\frac{\mathbf{c}_{(l)}}{\sigma_C} && \text{for } l=1:K-1 \\
& \mathbf{t}_{(l)}\ge\mathbf{0},\ \mathbf{c}_{(l)}\ge\mathbf{0} && \text{for } l=1:K
\end{aligned} \tag{4}$$

The problem (4) looks for the first time as much more complicated than problem (1), but contrary to (1), all the design variables are non-negative and all constraints are equalities, so after comparison of the standard form of (1) with (4), one can easily deduce that the size of the problem is reduced more than twice. Note that the form (4) is new in the literature and more economical than the formulations proposed in [3] or [7]. Moreover, for $K=1$ it automatically reduces to a well-known form used for one-load case problem (see [1] or [5]) thus no additional separate code is needed for this special case.
Note that (4) is written directly in the standard linear programming formulation

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \mathbf{c}^T\mathbf{x} \\
s.t. \quad & \mathbf{H}\mathbf{x}=\mathbf{b} \\
& \mathbf{x}\ge\mathbf{0}
\end{aligned} \tag{5}$$

but the order of the design variables $\mathbf{t}_{(l)}$ and $\mathbf{c}_{(l)}$ is very important because it influences the bandwidth of the coefficient matrix $\mathbf{H}$ and consequently the computational time. The best choice is grouping the variables by subsequent load cases:

$$\mathbf{x}_{2KM}=\{\mathbf{t}_{(1)},\mathbf{c}_{(1)},\mathbf{t}_{(2)},\mathbf{c}_{(2)},\ldots,\mathbf{t}_{(K)},\mathbf{c}_{(K)}\} \tag{6}$$

so the appropriate cost vector $\mathbf{c}$ and right hand side vector $\mathbf{b}$ can be defined as

$$\begin{aligned}
\mathbf{c}_{2KM} &= \{\mathbf{L}/\sigma_T,\mathbf{L}/\sigma_T,\mathbf{0},\ldots,\mathbf{0}\} \\
\mathbf{b}_{KN+(K-1)M} &= \{\mathbf{P}_{(1)},\mathbf{0},\mathbf{P}_{(2)},\mathbf{0},\ldots,\mathbf{P}_{(K)}\}
\end{aligned} \tag{7}$$

The coefficient matrix of problem (23) under ordering (25) has a possibly small bandwidth (almost block-diagonal) and as before (c.f. (7)) is regular, repetitive, very sparse; and can be written as

$$\mathbf{H}_{(KN+(K-1)M)\,x\,2KM}=\begin{bmatrix}\mathbf{F} & \mathbf{0} & \mathbf{0} & & \mathbf{0}\\ \mathbf{G} & -\mathbf{G} & \mathbf{0} & & \mathbf{0}\\ \mathbf{0} & \mathbf{F} & \mathbf{0} & \dots & \mathbf{0}\\ \mathbf{0} & \mathbf{G} & -\mathbf{G} & & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{F} & & \mathbf{0}\\ & \vdots & \ddots & \ddots & \vdots\\ & \vdots & & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{G}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{F}\end{bmatrix} \tag{8}$$

where sub-matrices **F** and **G** are defined as follows

$$\mathbf{F}_{N\times 2M}=\begin{bmatrix}\mathbf{B^T} & -\mathbf{B^T}\end{bmatrix} \quad \text{and} \quad \mathbf{G}_{M\,x\,2M}=\begin{bmatrix}\mathbf{I}/\sigma_T & \mathbf{I}/\sigma_C\end{bmatrix} \tag{9}$$

The evident advantage of the formulation (4) can be recognized after comparing the sizes of coefficient matrices **H** in (1) and (4) and one can easy check that the size (both number of rows and columns) of matrix **H** in (4) is more than two times smaller. Moreover, the new matrix **H** is more efficient for computation due to its almost block-diagonal form, and for rather special case with only one load condition $\mathbf{H}=\mathbf{F}$ which means that it is as efficient as casual formulation for one-load case problem (c.f. Sokół [5]).

For activating new bars in the adaptive ground structure method we need also dual variables but using the *primal-dual version of the interior point method* they are calculated automatically and for free. The convenient for next treatment dual form of multi-load case problem was derived in [7] and is given by

$$\begin{aligned}&\max_{\mathbf{u}_{(l)}\in\mathbf{R}^N,\mathbf{e}^+_{(l)},\mathbf{e}^-_{(l)}\in\mathbf{R}^M} W=\sum_l \mathbf{P}^T_{(l)}\mathbf{u}_{(l)}\\ &s.t.\quad \sum_l(\sigma_T\mathbf{e}^+_{(l)}+\sigma_C\mathbf{e}^-_{(l)})\le\mathbf{L}\\ &\qquad \mathbf{B}\,\mathbf{u}_{(l)}-\mathbf{e}^+_{(l)}\le\mathbf{0}\\ &\qquad -\mathbf{B}\,\mathbf{u}_{(l)}-\mathbf{e}^-_{(l)}\le\mathbf{0}\\ &\qquad \mathbf{e}^+_{(l)}\ge\mathbf{0},\quad \mathbf{e}^-_{(l)}\ge\mathbf{0}\\ &\qquad \text{for}\quad l=1,2,\dots,K\end{aligned} \tag{10}$$

where **u**, $\mathbf{e}^+$, $\mathbf{e}^-$ are Lagrange multipliers, called *adjoint nodal displacements* and *adjoint member elongations* for tension and compression, respectively. They are independent variables for every load condition (l) but constrained together by $(10)_2$ which enables deriving the generalized optimality criteria for multi-load trusses. These criteria can be formulated as follows (see [7] for details).

Theorem

In the stress-based multi-load truss optimization problem the optimal solution has to satisfy the following conditions:

1) for every bar of the truss the "*total adjoint multi-load strain* $\hat{\varepsilon}_i$ " is restricted by 1

$$\forall i=1,2,\dots,M \qquad \hat{\varepsilon}_i\le 1 \tag{11}$$

where

$$\hat{\varepsilon}_i=\sum_l(\sigma_T\varepsilon^+_{(l),i}+\sigma_C\varepsilon^-_{(l),i}) \tag{12}$$

$$\begin{aligned}\varepsilon^+_{(l),i}&=\max(\varepsilon_{(l),i},0)\\ \varepsilon^-_{(l),i}&=\max(-\varepsilon_{(l),i},0)\end{aligned} \tag{13}$$

$$\varepsilon_{(l),i}=\mathbf{B}_i\,\mathbf{u}_{(l)}/L_i \tag{14}$$

2) moreover, the non-zero cross-section area A_i is needed only for '*fully strained*' bar:

$$\begin{aligned} &\text{if } \ A_i > 0 \ \text{ then } \ \hat{\varepsilon}_i = 1 \\ &\text{if } \ \hat{\varepsilon}_i < 1 \ \text{ then } \ A_i = 0 \end{aligned} \tag{15}$$

The term "*fully strained*" corresponds to the total and normalized adjoint strain defined in (12) but from the other point of view every bar in the truss is also fully stressed for some load case but not necessarily for other ones. It is a subtle and different situation from one-load case problem for which all bars are fully stressed together. Here, for multi-load case, some bars can be fully stressed only for chosen load conditions and can even be inactive for other ones.

The conditions (11,12) define the domain of feasible adjoint strain fields and can be utilized in the adaptive ground structure method discussed in the next section. They are used to filter the set of new active bars and also as a stop criterion.

5. The adaptive ground structure method with selective subsets of active bars and nodes

Due to limited space of the paper we can describe the new method only briefly. The main idea of activating new bars is the same as before [7] but now after each iteration the nodes are split into two subsets: active and inactive nodes. Then, in the subsequent iteration the adjoint displacements are updated only for active nodes. Inactive nodes appear in empty regions where no material is needed. Thus before starting the next iteration all bars connected with inactive nodes can also be eliminated (temporary for the current iteration). Consequently, the size of the coefficient matrix of the problem (3) is much smaller in terms of the number of rows and columns. It should be noted that inactive nodes are not removed forever from the ground structure and can be activated if necessary. Moreover, the adjoint displacements of these nodes have to be preserved for subsequent iterations for checking the optimality criteria. Of course this complicates the code but is necessary and crucial to assure convergence to a globally optimal solution. The step by step procedure for the proposed method can be described as follows:

First iteration:

1. Set *iter* = 1, $d = 1$ and generate the initial ground structure $N_x{\times}N_y{\times}N_z$:1×1×1 with bars connecting only the neighbouring nodes. Contrary to the previous versions these bars can also be deactivated in subsequent iterations.
2. Solve the problem (4) for this initial ground structure and get the dual variables $\mathbf{u}_{(l)}^{(1)}$.

Next iterations:

3. Increment the number of iteration *++iter*.
4. Increment the distance of connections $d := \max(d_{max}, d{+}1)$, together with d_x, d_y, d_z.
5. Select the new set of active bars in the ground structure $N_x{\times}N_y{\times}N_z{:}d_x{\times}d_y{\times}d_z$:
   - for every bar compute normalized strain using the displacement fields from the previous iteration: $\mathbf{u}_{(l)}^{(iter-1)} \;\Rightarrow\; \hat{\varepsilon}_i = \sum_l (\sigma_T \varepsilon^{+}_{(l),i} + \sigma_C \varepsilon^{-}_{(l),i})$ (see (12)-(13)),
   - if $\hat{\varepsilon}_i \geq 1 - tol$, then activate (add) *i*-th bar,
   - otherwise, if $\hat{\varepsilon}_i < 0.3$ and $d < d_{max}$ then deactivate (remove) bar,
   - if $d < d_{max}$ and the number of added bars is too small then go to step 4.
6. Check the stopping criterion:
   - if $d = d_{max}$ and there are no new bars added then finish (we approach the optimum solution because for all potential bars $i = 1{:}M$ the constraints $(10)_2$ are satisfied and the solution cannot be further improved)
7. Calculate the volumes of material connected to nodes; if the volume of a chosen node is equal or close to zero and, moreover, no any new bar is added to this node, then deactivate it together with all connecting bars; then remove the appropriate degrees of freedom from the system but keep adjoint displacements of inactive nodes 'frozen' for the next iteration.
8. Solve primal problem (4) for reduced system of active bars and nodes and get dual variables $\mathbf{u}_{(l)}^{(iter)}$ (combine the updated adjoint displacements of active nodes with frozen displacements of inactive nodes).
9. Repeat from step 3.

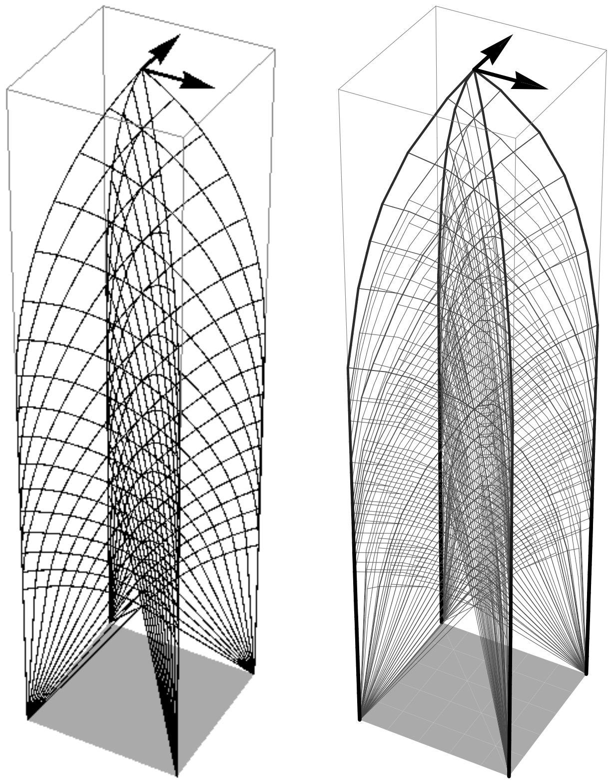
The program implementing the above algorithm has been written in Mathematica using parallel computing.

6. Examples of two-load case problems with spatial Michell trusses

In both examples presented in this section we assume: a) equal permissible stresses in tension and compression $\sigma_T = \sigma_C = \sigma_0$, b) and equal magnitudes of applied point forces $\|\mathbf{P}_{(1)}\| = \|\mathbf{P}_{(2)}\| = P$.

As the first example, consider the two-load case problem presented in Fig. 1 in which the two independent point loads are applied in the centre of the upper square of the bounding cuboidal domain $d/\sqrt{2} \times d/\sqrt{2} \times 3d$ and

directed along *x* and *y* axes, while the continuous full support is applied on the whole bottom square.
In Fig. 1a we present the exact optimal solution obtained using the superposition principles and the concept of component loads [4]. The optimal structure is composed of two orthogonal long cantilevers lying in diagonal planes. The exact volume of the structure can be calculated using the formulae derived by Lewiński et al. [2]. The numerical confirmation of this analytical prediction is presented in Fig. 1b and was performed for the ground structure with 20×20×60 cells, 26901 nodes and more than 300mln potential bars. The solution was obtained in less than 2 hours using classical computer with Intel i7 processor which clearly indicates a good efficiency of the proposed method. Note that 'numerical' volume is only 0.3% worse than the exact analytical solution.



$V_{\text{exact}} = 19.22932421\, P\, d / \sigma_0$ $V_{\text{num}} = 19.2818\, P\, d / \sigma_0$

Figure 1: Example of a 3D Michell structure for two loading conditions:
a) an exact analytical prediction using the concept of component loads,
b) numerical confirmation using the ground structure with 300mln potential bars

The second example is presented in Fig. 2 and is a subtle modification of the previous problem. Now the independent point loads are directed along the diagonals of the upper square of the bounding cuboidal domain $d \times d \times 3d$. Moreover, these loads have to be optimally transmitted to four fixed supports in the corners of bottom square. The exact analytical solution of this modified problem is harder to predict even using the concept of component loads. Hence in this example we firstly discovered the optimal layout numerically using the same density of the ground structure as before (i.e. more than 300mln bars). The numerical result presented in Fig. 2a suggests that the exact solution consists of four plane long cantilevers forming a specific hip roof (it is not clearly visible just from Fig. 2a but is evident after rotating this structure in 3D space). Then, employing the layout predicted numerically, the new exact analytical solution was obtained and presented in Fig. 2b. As before, the exact volume of this complex structure can be calculated using the formulae derived in [2], and is about 0.3% better than the volume obtained numerically. At the end, both numerical and analytical solutions of Fig. 2ab were also verified by using the superposition of two trusses corresponding to appropriate component loads, see Fig. 3c.

7. Concluding Remarks

Note that the optimal solutions presented in Figs 1 and 2 form shell-like structures composed of lattice surfaces with Michell trusses inside. Hence most of design space is empty and it was the main motivation for developing a new improved method in which both bars and nodes can be eliminated if unnecessary. As before, despite the iterative nature of the method the convergence to a global optimum as guaranteed.

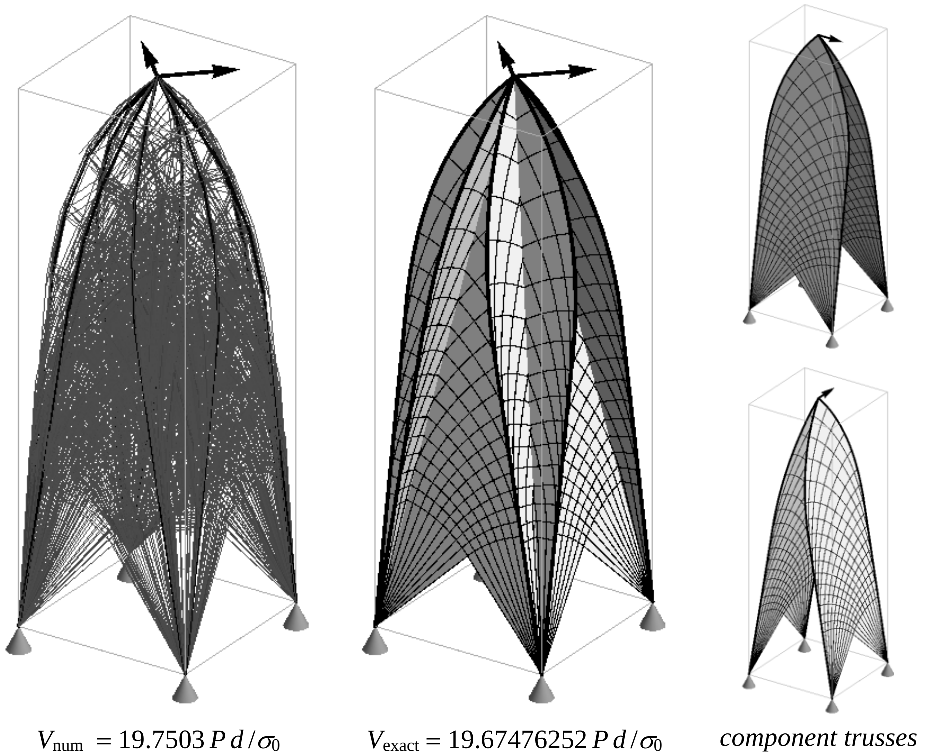


Figure 2: Example of 3D Michell truss for two independent load cases: a) numerical recognition of the optimal layout; b) exact analytical solution; c) optimal trusses for two component loads

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